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RESEARCH TITLE

A PERTURBATION THEORY FOR THE SEMIGROUP OF OPERATORS ON HILBERT SPACES OF SEQUENCES

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Abstract

In this paper we fixate a perturbation result for C_0-semigroups on Hilbert spaces and use it to exhibit that certain sequence of operators of the form $A_j u^j = i(u^j)^{(2k)} + V \cdot (u^j)^l$ on $L^2(\mathbb{R})$ generates a semigroup that is strongly continuous on $(0,\infty)$. Applications of C_0-semigroup perturbation theory are crucial for solving differential equations.

Key Words: C_0- Semigroup, A perturbation Theory, Sequence Operators, Hilbert Spaces

1. Introduction

A minimal condition in several of the known perturbation theorems is the relative boundedness of the perturbation of sequence B_j in terms of the given semigroup sequence of generators A_j. These relative boundedness requirements are typically described as

$$\left\|\sum_j B_j(\lambda_j - A_j)^{-1}\right\| \leq M < 1 \quad (1)$$

or

$$\left\|\sum_j (\lambda_j - A_j)^{-1} B_j x^j\right\| \leq M\left\|\sum_j x^j\right\| \quad (2)$$

on a certain subset of the complex plane. E.g., in the proof of the well-known result for bounded perturbations (see e.g. [5, Chapter III, Theorem 1.3], [7, Chapter 3, Theorem 1.1]) condition (1) is one of the main ideas. The Miyadera-Voigt, respectively Desch-Schappacher, perturbation theorem uses (1), respectively (2) (see [5, Chapter III, Section 3]). If A_j sequence of generates a bounded analytic semigroup, therefore condition (1), satisfied for every λ_j in the right half plane, is sufficient to exhibit that $A_j + B_j$ again the sequence of generates an analytic semigroup. Obviously, this cannot be true for general C_0-semigroups. But in this article we want to explore what can be said about $A_j + B_j$ if we only suppose the relative boundedness conditions (1) and (2) on a half plane. If the underlying space is a Hilbert space, we can exhibit that $\left(A_j + B_j, D\left(A_j\right)\right)$ the sequence of generates a semigroup that is strongly continuous on $(0, \infty)$.

This article is organized as follows. In the second section we collect some facts about semigroups that are strongly continuous on $(0, \infty)$. Section 3 contains the main results which are showed in Sections 4 and 5. In Section 6 we apply the theorem to certain differential operators.

2. Semigroups that are strongly continuous on $(0, \infty)$

Let X be a Banach space. By $\mathcal{L}(X)$ we denote the Banach space of each bounded linear sequence of operators from X to X. If $T^j: (0, \infty) \rightarrow \mathcal{L}(X)$ is a strongly continuous mapping (i.e., $(1 + \varepsilon) \rightarrow T^j(1 + \varepsilon)x^j$ is continuous on $(0, \infty)$ for each $x^j \in X$) that satisfies the semigroup property $T^j(1 + \varepsilon)T^j(1 + \varepsilon) = T^j(2 + 2\varepsilon)$ for all $\varepsilon > -1$, therefore we say that the families $\left(T^j(1+\varepsilon)\right)_{\varepsilon>-1}$ is a semigroup that is strongly continuous on $(0,\infty)$. Examples for such semigroups can be found in [3], [6, Section I.8] and [5, Chapter I, 5.9 (7)].

In this article we want to use Laplace transform methods. Therefore we will suppose from now on that the mapping T^j is locally integrable on $(0,\infty)$ (i.e., $T^j \in L^1\left(\left(0,(1+\varepsilon)\right);\mathcal{L}(X)\right)$ for every $\varepsilon>-1$) and

$$\left\|\sum_j \int_0^{(1+\varepsilon)} T^j(1+\varepsilon)d(1+\varepsilon)\right\| \leq M\sum_j e^{\omega^j(1+\varepsilon)} \qquad \varepsilon>-1, \qquad (3)$$

for some constants M and ω^j. Therefore, due to [2, Proposition 1.4.5], we can define the Laplace transform for $\lambda_j>\omega^j$. Using integration by parts and the semigroup property, we find that $\left(R(\lambda_j)\right)_{\lambda_j>\omega^j}$ satisfies the resolvent equation $R(\lambda_j)-R(\mu_j)=(\mu_j-\lambda_j)R(\lambda_j)R(\mu_j)$. Therefore the following definition makes sense.

Definition 2.1. Let $\left(T^j(1+\varepsilon)\right)_{\varepsilon>-1}$ be a semigroup on a Banach space X that is strongly continuous and locally integrable on $(0,\infty)$ and satisfies the norm estimate (3). If there exists a linear sequence of operators $(A_j, D(A_j))$ in X, where $D(A_j)\subseteq X$ is the domain of A_j, such that (ω^j,∞) is contained in the resolvent set $\rho(A_j)$ of A_j and

$\sum_j R(\lambda_j, A_j) := \sum_j (\lambda_j I - A_j)^{-1} = \int_0^\infty \sum_j e^{\lambda_j(1+\varepsilon)} T^j(1+\varepsilon)d(1+\varepsilon),\ \lambda_j>\omega^j,$

therefore $(A_j, D(A_j))$ are called the sequence of generators of $\left(T^j(1+\varepsilon)\right)_{\varepsilon>-1}$.

Using this definition, one can exhibit easily the following properties of the semigroup $\left(T^j(1+\varepsilon)\right)_{\varepsilon>-1}$ and its sequence of generator A_j:

(a) if $x^j \in D(A_j)$, therefore $T^j(1+\varepsilon)x^j \in D(A_j)$ and $A_j T^j(1+\varepsilon)x^j = T^j(1+\varepsilon)A_j x^j$ for every $\varepsilon>-1$,

(b) if $x^j \in D(A_j)$ and $\varepsilon>-1$, therefore

$$\sum_j x^j = \sum_j T^j(1+\varepsilon)x^j - \int_0^{(1+\varepsilon)} \sum_j T^j(1+\varepsilon)A_j x^j d(1+\varepsilon).$$

The properties (a) and (b) imply that for $x^j \in D(A_j)$ the function u_x^j, defined by

$u_x^j(1+\varepsilon) = T^j(1+\varepsilon)x^j$ $(\varepsilon>-1)$ and $u_x^j(0)=x^j$, is a solution of the abstract Cauchy problem

$$\begin{cases} (u^j)' = A_j u^j(1+\varepsilon), \ \varepsilon > -1, \\ u^j(0) = x^j. \end{cases} \tag{4}$$

Here, by a solution of (4) we mean a function $u^j \in C([0,\infty);X) \cap C^1((0,\infty);X)$ such that $u^j(1+\varepsilon) \in D(A_j)$ and $(u^j)'(1+\varepsilon) = A_j u^j(1+\varepsilon)$ for every $\varepsilon > -1$ and $u^j(0) = x^j$

3. Main result

The main result is the following perturbation theorem for C_0-semigroups on Hilbert spaces.

Theorem 3.1. Let $\left(A_j, D(A_j)\right)$ be the sequence of generators of a C_0-semigroup $\left(T^j(1+\varepsilon)\right)_{\varepsilon \geq -1}$ on a Hilbert space X and let $\left(B_j, D(B_j)\right)$ be a closed sequence of operators in X such that $D(B_j) \supseteq D(A_j)$. We suppose that there exist constants $0 \leq M < 1$ and $(\lambda_j)_0 \in \mathbb{R}$ such that the set $\left\{\lambda_j \in \mathbb{C} : Re\lambda_j \geq (\lambda_j)_0\right\}$ is contained in the resolvent set of A_j and the estimates

$$\left\|\sum_j B_j R(\lambda_j, A_j) x^j\right\| \leq M \left\|\sum_j x^j\right\| \tag{5}$$

and

$$\left\|\sum_j R(\lambda_j, A_j) B_j y^j\right\| \leq M \left\|\sum_j y^j\right\| \tag{6}$$

are satisfied for all $\lambda_j \in \mathbb{C}$ with $\operatorname{Re} \lambda_j \geq (\lambda_j)_0$ and all $x^j \in X$, $y^j \in D(B_j)$. Therefore $\left(A_j + B_j, D(A_j)\right)$ the sequence of generates a semigroup $(S(1+\varepsilon))_{\varepsilon \geq -1}$ that are strongly continuous on $(0,\infty)$.

Example 3.2. Suppose $X = L^2(0,\infty)$. Define linear sequences of operators $\left(A_j, D(A_j)\right)$ and $\left(B_j, D(B_j)\right)$ by $\left(A_j f_j\right)(x^j) := (f_j)'(x^j)$ and $\left(B_j f_j\right)(x^j) := \frac{1}{3x^j} f_j(x^j)$ with maximal domains. Using Hardy's Inequality, we can exhibit that $\left\|\sum_j R(\lambda_j, A_j) B_j x^j\right\|_2 \leq \frac{2}{3} \left\|\sum_j x^j\right\|_2$ for all $Re\, \lambda_j > 0$, i.e. condition (6) is satisfied. The "candidate" for the perturbed semigroup is $S(1+\varepsilon) f_j(x^j) := (x^j)^{-1/3} \left(x^j + (1+\varepsilon)\right)^{1/3} f_j\left(x^j + (1+\varepsilon)\right)$. But $S(1+\varepsilon)$ is not a bounded sequence of operators on X.

To fixate Theorem 3.1 we shall use the following result about the sequence of generators for semigroups that are strongly continuous on $(0,\infty)$.

Theorem 3.3. Let $\left(A_j, D(A_j)\right)$ be a closed, densely defined the sequence of operators on a Hilbert space X such that the resolvent $R(\lambda_j, A_j)$ exists and is uniformly bounded on $\{\lambda_j \in \mathbb{C} : Re\lambda_j \geq 0\}$. Further, we suppose that there exists a constant $C \geq 0$ such that

$$\left(\int_{-\infty}^{\infty} \left\|\sum_j R\left(i(1+\varepsilon), A_j\right)x^j\right\|^2 d(1+\varepsilon)\right)^{1/2} \leq C\left\|\sum_j x^j\right\| \tag{7}$$

and

$$\left(\int_{-\infty}^{\infty} \left\|\sum_j R\left(i(1+\varepsilon), \left(A_j\right)^*\right)x^j\right\|^2 d(1+\varepsilon)\right)^{1/2} \leq C\left\|\sum_j x^j\right\| \tag{8}$$

For every $x^j \in X$. Therefore $\left(A_j, D(A_j)\right)$ the sequence of generates a semigroup $\left(T^j(1+\varepsilon)\right)_{\varepsilon \geq -1}$ that is strongly continuous on $(0,\infty)$.

Example 3.4. We regard the space $X = L^2(\mathbb{R}) \times L^2(\mathbb{R})$ which is a Hilbert space if we choose the norm $\left\|\sum_j (u^j, v^j)\right\|_X := \left(\left\|\sum_j u^j\right\|_2^2 + \left\|\sum_j v^j\right\|_2^2\right)^{1/2}$. For $k \in \mathbb{N}$ and $0 \leq (\varepsilon+\beta) < 4k$ we define the function $(1+\varepsilon): \mathbb{R} \to \mathbb{R}^2$ by

$$(1+\varepsilon)x^j = \begin{pmatrix} -1-\left(x^j\right)^{2k} & \left(x^j\right)^{(\varepsilon+\beta)} \\ 0 & -1-\left(x^j\right)^{2k} \end{pmatrix} \tag{9}$$

Therefore the multiplication sequence of operators, given by

$$A_j\left(u^j, v^j\right) = (1+\varepsilon)\begin{pmatrix} u^j \\ v^j \end{pmatrix} \quad , D(A_j) = \left\{(u^j, v^j) \in X : A_j(u^j, v^j) \in X\right\}, \tag{10}$$

satisfies the conditions of Theorem 3.3, Therefore A_j the sequence of generates a semigroup that are strongly continuous on $(0,\infty)$. But if $k < \frac{(\varepsilon+\beta)}{2} < 2k$, therefore A_j is not strongly continuous at 0.

Note

We can deduce that:

(i) $\left(\int_{-\infty}^{\infty} \left\|\sum_j R\left(i(1+\varepsilon), A_j\right)x^j\right\|^2 d(1+\varepsilon)\right)^{1/2} \leq \left(\int_{-\infty}^{\infty} \left\|\sum_j R\left(i(1+\varepsilon), \left(A_j\right)^*\right)x^j\right\|^2 d(1+\varepsilon)\right)^{1/2}$

(ii) $\left(\int_{-\infty}^{\infty} \left\|\sum_j R\left(i(1+\varepsilon), A_j\right)x^j\right\|^2 d(1+\varepsilon)\right)^{1/2} \leq \frac{C}{M}\left\|\sum_j B_j R(\lambda_j, A_j)x^j\right\|$

(iii) $\left(\int_{-\infty}^{\infty}\left\|\sum_j R\left(i(1+\varepsilon),\left(A_j\right)^*\right)x^j\right\|^2 d(1+\varepsilon)\right)^{1/2} \leq \frac{C}{M}\left\|\sum_j B_j R(\lambda_j, A_j)x^j\right\|$

Proof

(i) From (7) and (8).

(ii) From (5) and (7).

(iii) From (5) and (8).

4. Proof of Theorem 3.3

We give a proof of Theorem 3.3. We first state two technical lemmas.

Lemma 4.1. Let $\left(A_j, D\left(A_j\right)\right)$ be a closed sequence of operators in a Banach space X with

$0 \in \rho(A_j)$. If we can find a subset G of $\rho(A_j)$ and a constant $M \geq 0$ such that $\left\|\sum_j R(\lambda_j, A_j)\right\| \leq M$ on G, then there is a constant $c \geq 0$ such that

$\left\|\sum_j R(\lambda_j, A_j)x^j\right\| \leq \frac{c}{1+\left\|\sum_j \lambda_j\right\|}\left\|\sum_j A_j x^j\right\|$ and $\left\|\sum_j R(\lambda_j, A_j)^2 y^j\right\| \leq \frac{c}{1+\left\|\sum_j \lambda_j\right\|^2}\left\|\sum_j \left(A_j\right)^2 y^j\right\|$

for all $\lambda_j \in G$ and all $x^j \in D(A_j)$, $y^j \in D(A_j)^2$.

Proof. For $\lambda_j \in G\backslash\{0\}$ and $x^j \in D(A_j)$ the resolvent $R(\lambda_j, A_j)x^j$ can be written as $R\left(\lambda_j, A_j\right)x^j = \frac{1}{\lambda_j}\left(x^j + R\left(\lambda_j, A_j\right)A_j x^j\right)$. If $y^j \in D(A_j)^2$ we obtain $R(\lambda_j, A_j)^2 y^j = \frac{1}{\left(\lambda_j\right)^2}\left(y^j + 2R\left(\lambda_j, A_j\right)A_j y^j + R\left(\lambda_j, A_j\right)^2\left(A_j\right)^2 y^j\right)$. Since 0 is in the resolvent set of A_j and the resolvent are uniformly bounded on G, the lemma is proved.

Lemma 4.2. Let $\left(A_j, D\left(A_j\right)\right)$ be a closed sequence of operators in a Banach space X such that $\left\{\lambda_j \in \mathbb{C} : Re\lambda_j \geq 0\right\} \subseteq \rho(A_j)$ and $\left\|\sum_j R(\lambda_j, A_j)\right\| \leq M$ for all $\lambda_j \in \mathbb{C}$ with $Re\,\lambda_j \geq 0$. For $x^j \in X$, $\varepsilon \geq -1$ we define

$$U(1+\varepsilon)\sum_j x^j := \frac{1}{2\pi i(1+\varepsilon)}\int_{(1+\varepsilon)-i\infty}^{(1+\varepsilon)+i\infty}\sum_j e^{\mu_j(1+\varepsilon)}R\left(\mu_j, A_j\right)^2 x^j d\mu_j \qquad (11)$$

Therefore ,

(a) if $x^j \in D(A_j)^2$, the integral in (11) are certainly convergent and does not depend on $\varepsilon \geq -1$,

(b) for every $x^j \in D(A_j)^2$ and all $\varepsilon > -1$, the limit

$$\lim_{\varepsilon\to\infty}\frac{1}{2\pi i}\int_{(1+\varepsilon)-i(1+\varepsilon)}^{(1+\varepsilon)+i(1+\varepsilon)}\sum_j e^{\mu_j(1+\varepsilon)}R(\mu_j,A_j)x^j d\,\mu_j \qquad (12)$$

exists and are equals to $U(1+\varepsilon)x^j$,

(c) for $x^j \in D(A_j)^2$ and $Re\lambda_j > 0$, we have that

$$\sum_j R(\lambda_j,A_j)x^j = \sum_j \frac{x^j}{\lambda_j} + \int_0^\infty \sum_j e^{\lambda_j(1+\varepsilon)}\left(U(1+\varepsilon)x^j - x^j\right) d\,(1+\varepsilon), \quad (13)$$

(d) the semigroup property

$$U(1+\varepsilon)U(1+\varepsilon)x^j = U(2+2\varepsilon)x^j$$

holds for all $\varepsilon > -1$ and every $x^j \in D(A_j)^4$.

Proof. Let $x^j \in D(A_j)^2$ and $\varepsilon > -1$.

(a) Lemma 4.1 implies that the integral in (11) converges absolutely. The independence of $\varepsilon \geq -1$ is a consequence of Cauchy's Theorem.

(b) Integration by parts yields that for $\varepsilon > -1$

$$\int_{(1+\varepsilon)-i(1+\varepsilon)}^{(1+\varepsilon)+i(1+\varepsilon)} \sum_j e^{\mu_j(1+\varepsilon)}R(\mu_j,A_j)x^j d\,\mu_j$$

$$= \frac{1}{(1+\varepsilon)}\sum_j \left(e^{(1+\varepsilon)+i(1+\varepsilon)^2} R\begin{pmatrix}(1+\varepsilon)+\\ i(1+\varepsilon),A_j\end{pmatrix}x^j\right) -$$

$$e^{(1+\varepsilon)-i(1+\varepsilon)^2} R\left((1+\varepsilon)-i(1+\varepsilon),A_j\right)x^j\Big) +$$

$$\frac{1}{(1+\varepsilon)}\int_{(1+\varepsilon)-i(1+\varepsilon)}^{(1+\varepsilon)+i(1+\varepsilon)}\sum_j e^{\mu_j(1+\varepsilon)}R\left(\mu^j,A_j\right)^2 x^j d\,\mu_j.$$

By Lemma 4.1, $\left\|\sum_j R\left(i(1+\varepsilon),A_j\right)x^j\right\|$ converges to 0 if $\|1+\varepsilon\| \to \infty$. Therefore we have that the limit 12 exists and are equals to $U(1+\varepsilon)x^j$.

(c) Let $Re\lambda_j > 0$. If $x^j \in D(A_j)$, $\varepsilon > -1$ and $0 < (1+\varepsilon) < Re\lambda_j$, we find

$$\sum_j \left(U(1+\varepsilon)x^j - x^j\right) = \frac{1}{2\pi i}\int_{(1+\varepsilon)-i\infty}^{(1+\varepsilon)+i\infty}\sum_j e^{\mu_j(1+\varepsilon)}\left(R(\mu_j,A_j)x^j - \frac{x^j}{\mu_j}\right)d\,\mu_j$$

$$= \frac{1}{2\pi i}\int_{(1+\varepsilon)-i\infty}^{(1+\varepsilon)+i\infty}\sum_j e^{\mu_j(1+\varepsilon)}\left(R(\mu_j,A_j)A_j x^j\right)\frac{d\mu_j}{\mu_j}$$

For $x^j \in D(A_j)^2$, Lemma 4.1 yields

$\left\|\sum_j\left(R(\mu_j,A_j)A_j x^j\right)\right\| \leq \frac{c}{1+\left|\sum_j \mu_j\right|}\left\|\sum_j (A_j)^2 x^j\right\|$. Therefore the above integral is absolutely convergent and $\left\|\sum_j U(1+\varepsilon)x^j - x^j\right\| \leq c'\left\|\sum_j (A_j)^2 x^j\right\|$ for every $\varepsilon >$

-1. So we can form the Laplace transform of $U(1+\varepsilon)x^j - x^j$ and obtain

$$\sum_j \left(\lambda_j \int_0^\infty e^{-\lambda_j(1+\varepsilon)}\big(U(1+\varepsilon)x^j - x^j\big)d\,(1+\varepsilon)\right)$$

$$= \sum_j \left(\frac{\lambda_j}{2\pi i}\int_0^\infty e^{-\lambda_j(1+\varepsilon)} \int_{(1+\varepsilon)-i\infty}^{(1+\varepsilon)+i\infty} e^{\mu_j(1+\varepsilon)} R(\mu_j, A_j)A_j x^j \frac{d\mu_j}{\mu_j} d(1+\varepsilon)\right)$$

$$= \sum_j \left(\frac{\lambda_j}{2\pi i}\int_{(1+\varepsilon)-i\infty}^{(1+\varepsilon)+i\infty}\int_0^\infty e^{(\mu_j-\lambda_j)(1+\varepsilon)} d(1+\varepsilon) R(\mu_j, A_j)A_j x^j \frac{d\mu_j}{\mu_j}\right)$$

$$= \sum_j \left(\frac{\lambda_j}{2\pi i}\int_{(1+\varepsilon)-i\infty}^{(1+\varepsilon)+i\infty} \frac{1}{\lambda_j-\mu_j} R(\mu_j, A_j)\, A_j x^j \frac{d\mu_j}{\mu_j}\right)$$

$$= \sum_j R(\lambda_j, A_j)A_j x^j = \sum_j \big(\lambda_j R(\lambda_j, A_j)x^j - x^j\big),$$

using Fubini's and Cauchy's Theorems.

(d) Let $\mu^j > \lambda_j > 0$. Therefore integration by parts yields

$$\sum_j \left(\frac{R(\lambda_j, A_j)x^j - R\left(\mu_j, A_j\right)x^j}{\mu_j - \lambda_j}\right)$$

$$= \int_0^\infty \sum_j \left(e^{(\lambda_j-\mu_j)(1+\varepsilon)} R(\lambda_j, A_j)x^j d(1+\varepsilon) - \frac{x^j}{\mu_j\left(\mu_j-\lambda_j\right)}\right)$$

$$-\sum_j \left(\frac{1}{\mu^j-\lambda_j}\int_0^\infty e^{(\lambda_j-\mu_j)(1+\varepsilon)} e^{(-\lambda_j)(1+\varepsilon)}\,\big(U(1+\varepsilon)x^j - x^j\big)d(1+\varepsilon)\right)$$

$$= \left(\int_0^\infty \sum_j e^{(\lambda_j-\mu_j)(1+\varepsilon)} \frac{x^j}{\lambda_j} d(1+\varepsilon)\right.$$

$$\left. + \left(\int_0^\infty \sum_j e^{(\lambda_j-\mu_j)(1+\varepsilon)} d(1+\varepsilon)\right)\left(\int_0^\infty \sum_j e^{-\lambda_j(1+\varepsilon)}\,\big(U(1+\varepsilon)x^j - x^j\big)d(1+\varepsilon)\right)\right)$$

$$-\left(\sum_j \frac{x^j}{\mu_j\left(\mu_j-\lambda_j\right)}\right.$$

$$\left.-\left(\int_0^\infty \sum_j e^{(\lambda_j-\mu_j)(1+\varepsilon)} d(1+\varepsilon)\right)\left(\int_0^{(1+\varepsilon)} \sum_j e^{-\lambda_j(1+\varepsilon)}\left(U(1+\varepsilon)x^j-x^j\right) d(1+\varepsilon)\right)\right)$$

$$=\sum_j \left(\frac{x^j}{\lambda_j\left(\mu_j-\lambda_j\right)}-\frac{x^j}{\mu_j\left(\mu_j-\lambda_j\right)}\right.$$

$$\left.+\left(\int_0^\infty \sum_j e^{(\lambda_j-\mu_j)(1+\varepsilon)} d(1+\varepsilon)\right)\left(\int_{(1+\varepsilon)}^\infty \sum_j e^{-\lambda_j(1+\varepsilon)}\left(U(1+\varepsilon)x^j-x^j\right)d(1+\varepsilon)\right)\right)$$

$$=\sum_j \left(\frac{\mu_j x^j-\lambda_j x^j}{\lambda_j\mu_j\left(\mu_j-\lambda_j\right)}\right.$$

$$\left.+\left(\int_0^\infty \sum_j e^{-\mu_j(1+\varepsilon)} d(1+\varepsilon)\right)\left(\int_{(1+\varepsilon)}^\infty \sum_j e^{\lambda_j(0)}\left(U(1+\varepsilon)x^j-x^j\right)d(1+\varepsilon)\right)\right)$$

$$=\left(\sum_j \frac{x^j}{\lambda_j\mu_j}+\left(\int_0^\infty \sum_j e^{-\mu_j(1+\varepsilon)} d(1+\varepsilon)\right)\left(\int_0^\infty \sum_j e^{-\lambda_j(1+\varepsilon)}\left(U(2+2\varepsilon)x^j-x^j\right) d(1+\varepsilon)\right)\right)$$

Second hand, if $x^j \in D(A_j)^4$, therefore $U(1+\varepsilon)x^j \in D(A_j)^2$ and

$$\sum_j R\left(\mu_j, A_j\right) R\left(\lambda_j, A_j\right)x^j$$

$$=\sum_j \left(\frac{R\left(\lambda_j, A_j\right)x^j}{\mu_j}+\int_0^\infty \sum_j e^{-\mu_j(1+\varepsilon)}\left(U(1+\varepsilon)R\left(\lambda_j, A_j\right)x^j-R\left(\lambda_j, A_j\right)x^j\right)\right) d(1+\varepsilon)$$

$$=\sum_j\left(\frac{x^j}{\lambda_j\mu_j}+\frac{1}{\mu_j}\int_0^\infty e^{-\lambda_j(1+\varepsilon)}\left(U(1+\varepsilon)x^j-x^j\right)\right)d(1+\varepsilon)$$

$$+\int_0^\infty\sum_j e^{-\mu_j(1+\varepsilon)}\left(U(1+\varepsilon)\frac{x^j}{\lambda_j}-\frac{x^j}{\lambda_j}\right)d(1+\varepsilon)$$

$$+\left(\int_0^\infty\sum_j e^{-\mu_j(1+\varepsilon)}\,d(1+\varepsilon)\right)\left(\int_0^\infty\sum_j e^{-\lambda_j(1+\varepsilon)}\left(U(1+\varepsilon)\left(U(1+\varepsilon)x^j-x^j\right)\right)\right.$$

$$\left.-\left(U(1+\varepsilon)x^j-x^j\right)\right)d(1+\varepsilon)$$

$$\sum_j\frac{x^j}{\lambda_j\mu_j}+\left(\int_0^\infty\sum_j e^{-\mu_j(1+\varepsilon)}\,d(1+\varepsilon)\right)\left(\int_0^\infty\sum_j e^{-\lambda_j(1+\varepsilon)}\left(U(1+\varepsilon)\left(U(1+\varepsilon)x^j-x^j\right)\right)\right)d(1+\varepsilon)$$

By the uniqueness theorem for the Laplace transform we obtain that

$$U(2+2\varepsilon)x^j-x^j=U(1+\varepsilon)U(1+\varepsilon)x^j-x^j \tag{14}$$

for almost all $\varepsilon>-1$ and for all $x^j\in D(A_j)^4$. For fixed $(1+\varepsilon)$, the functions $(1+\varepsilon)\longmapsto U(2+2\varepsilon)x^j$ and $(1+\varepsilon)\longmapsto U(1+\varepsilon)U(1+\varepsilon)x^j$ both are continuous. So the equation (14) holds for every $\varepsilon>-1$. By exchanging the roles of $(1+\varepsilon)$ we obtain

$$U(2+2\varepsilon)x^j=U(1+\varepsilon)U(1+\varepsilon)x^j$$

for every $\varepsilon>-1$ and all $x^j\in D(A_j)^4$.

Proof of Theorem 3.3. Prove the theorem in four steps. Here, c is always an appropriate constant, and by $\langle\cdot,\cdot\rangle$ we denote the inner product on X.

Step 1: A_j "candidate" for the semigroup

Apply the inverse Fourier transform to $R(i\,\cdot,A_j)x^j\in L^2(\mathbb{R},X)$: Take $\varepsilon>-1$ and $x^j\in X$ and define

$T^j(1+\varepsilon)x^j:=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_j e^{i(1+\varepsilon)^2}R\left(i(1+\varepsilon),A_j\right)x^j d(1+\varepsilon)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\sum_j e^{\lambda_j(1+\varepsilon)}R\left(\lambda_j,A_j\right)x^j d\lambda_j$.

Since X is a Hilbert space, Plancherel's theorem yields $T^j(\cdot)x^j\in L^2\left((0,\infty),X\right)$and $\left(\int_0^\infty\left\|\sum_j T^j(1+\varepsilon)x^j\right\|^2 d(1+\varepsilon)\right)^{1/2}\le c\left\|\sum_j x^j\right\|$ for every $x^j\in X$. Clearly, T^j are linear in x^j, and from Lemma 4.2 (d) we know that the semigroup property $T^j(1+\varepsilon)T^j(1+\varepsilon)x^j = T^j(2+2\varepsilon)x^j$ are satisfied whenever $x^j \in D\left(A_j\right)^4$. and $\varepsilon > -1$.

Step 2: Boundedness of $T^j(1+\varepsilon)$

First we regard the adjoint sequence of operators $\left(A_j\right)^*$. As in step 1 we can exhibit that

$\left(T^j\right)^*(\cdot)x^j$, defined by

$$\sum_j \left(T^j\right)^*(1+\varepsilon)x^j = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_j e^{i(1+\varepsilon)^2} R\left(i(1+\varepsilon), \left(A_j\right)^*\right)x^j d(1+\varepsilon)$$

$$= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\sum_j e^{\lambda_j(1+\varepsilon)} R\left(\lambda_j, \left(A_j\right)^*\right)x^j d\lambda_j,\ \varepsilon > -1,$$

are in $L^2\big((0,\infty),X\big)$ for every $x^j \in X$ and $\left(\int_0^\infty \left\|\sum_j \left(T^j\right)^*(1+\varepsilon)x^j\right\|^2 d(1+\varepsilon)\right)^{1/2} \leq c\left\|\sum_j x^j\right\|$. It is easy to see that $\langle y^j, T^j(1+\varepsilon)x^j\rangle = \langle \left(T^j\right)^*(1+\varepsilon)y^j, x^j\rangle$ for $x^j, y^j \in X$ and $\varepsilon > -1$.

Now let $\varepsilon > -1$, $x^j \in D\left(A_j\right)^4$ and $y^j \in X$. Therefore

$$(1+\varepsilon)\sum_j \langle y^j, T^j(1+\varepsilon)x^j\rangle = \int_0^{(1+\varepsilon)} \sum_j \langle y^j, T^j(1+\varepsilon)x^j\rangle d\,(1+\varepsilon)$$

$$= \int_0^{(1+\varepsilon)} \sum_j \langle y^j, T^j(0)T^j(1+\varepsilon)x^j\rangle d\,(1+\varepsilon)$$

$$= \int_0^{(1+\varepsilon)} \sum_j \langle \left(T^j\right)^*(0)y^j, T^j(1+\varepsilon)x^j\rangle d\,(1+\varepsilon)$$

$$\leq \int_0^{(1+\varepsilon)} \left\|\sum_j \left(T^j\right)^*(0)y^j\right\| \left\|\sum_j T^j(1+\varepsilon)x^j\right\| d(1+\varepsilon)$$

and we can estimate

$$\int_0^{(1+\varepsilon)} \left\|\sum_j \left(T^j\right)^*(0)y^j\right\| \left\|\sum_j T^j(1+\varepsilon)x^j\right\| d(1+\varepsilon)$$

$$\leq \left(\int_0^{(1+\varepsilon)} \left\|\sum_j \left(T^j\right)^*(0)y^j\right\|^2 d(1+\varepsilon)\right)^{1/2} \left(\int_0^{(1+\varepsilon)} \left\|\sum_j T^j(1+\varepsilon)x^j\right\|^2 d(1+\varepsilon)\right)^{1/2}$$

$$\leq \left(\int_0^\infty \left\| \sum_j \left(T^j\right)^* (1+\varepsilon) y^j \right\|^2 d(1+\varepsilon) \right)^{1/2} \left(\int_0^\infty \left\| \sum_j T^j (1+\varepsilon) x^j \right\|^2 d(1+\varepsilon) \right)^{1/2}$$

$$\leq c \left\| \sum_j x^j \right\| \left\| \sum_j y^j \right\|.$$

This yields $\left\|\sum_j T^j(1+\varepsilon)x^j\right\| \leq \frac{c}{1+\varepsilon}\left\|\sum_j x^j\right\|$ for $x^j \in D\left(A_j\right)^4$. Since $(A_j, D(A_j))$ are densely defined and injective, $D\left(A_j\right)^4$ are dense in X. So we have showed that $T(1+\varepsilon) \in \mathcal{L}(X)$. Furthermore, the semigroup property $T^j(1+\varepsilon)T^j(1+\varepsilon) = T^j(2+2\varepsilon)$ are satisfied for every $\varepsilon > -1$.

Step 3: The generator of $\left(T^j(1+\varepsilon)\right)_{\varepsilon>-1}$

Let $Re\lambda_j > 0$. We want of prove that $\sum_j R\left(\lambda_j, A_j\right) = \int_0^\infty \sum_j e^{-\lambda_j(1+\varepsilon)} T^j(1+\varepsilon) d(1+\varepsilon)$.

In Lemma 4.2 (c) we have already proved that $\sum_j R\left(\lambda_j, A_j\right)x^j = \sum_j \frac{x^j}{\lambda_j} + \int_0^\infty \sum_j e^{-\lambda_j(1+\varepsilon)}\left(U(1+\varepsilon)x^j - x^j\right)d(1+\varepsilon)$ for every$x^j \in D\left(A_j\right)^2$. Since $\left(\int_0^\infty \left\|\sum_j T^j(1+\varepsilon)x^j\right\|^2 d(1+\varepsilon)\right)^{1/2} \leq c\left\|\sum_j x^j\right\|$ and $D\left(A_j\right)^2$ are dense in X, the assertion is proved.

Step 4: Strong continuity on $(0,\infty)$

Finally, we exhibit that $(1+\varepsilon) \mapsto T^j(1+\varepsilon)x^j$ are continuous on $(0,\infty)$ for every $x^j \in X$.

For $x^j \in D\left(A_j\right)^2$, Lemma 4.1 yields that $T^j(1+\varepsilon)x^j - x^j =$

$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \sum_j e^{\lambda_j(1+\varepsilon)} R\left(\lambda_j, A_j\right)A_j x^j \frac{d\lambda_j}{\lambda_j}$

converges absolutely and uniformly on compact intervals. Therefore $(1+\varepsilon) \mapsto T^j(1+\varepsilon)x^j$ are continuous on $[0,\infty)$ if $x^j \in D\left(A_j\right)^2$. Since $D\left(A_j\right)^2$ are dense in X and $(1+\varepsilon)T^j(1+\varepsilon)$ are uniformly bounded (Step 2), the mapping $(1+\varepsilon) \mapsto T^j(1+\varepsilon)x^j$ are continuous on $(0,\infty)$ for each $x^j \in X$. This proves the theorem.

5. Proof of Theorem 3.1

The following lemma is implicit in the standard presentations of perturbation theory

([5, Chapter III], [7, Chapter 3]).

Lemma 5.1. Let $\left(A_j, D(A_j)\right)$ and $\left(B_j, D(B_j)\right)$ be closed sequence of operators on a Banach space X where $(A_j) \subseteq D(B_j)$. Assume that A_j and $(A_j)^*$ are densely defined and that the resolvent set of A_j are nonempty. If there exists $0 \leq M < 1$ and $\emptyset \neq G \subseteq \rho(A_j)$ such that

$$\left\|\sum_j B_j R(\lambda_j, A_j) x^j\right\| \leq M \left\|\sum_j x^j\right\| \text{ for all } x^j \in X \text{ and all } \lambda_j \in G$$

and

$$\left\|\sum_j R(\lambda_j, A_j) B_j x^j\right\| \leq M \left\|\sum_j x^j\right\| \text{ for all } x^j \in D(B_j) \text{ and all } \lambda_j \in G, \qquad (15)$$

therefore the sequence of operators $\left(A_j + B_j, D(A_j)\right)$ are closed and $G \subseteq \rho(A_j + B_j)$. Moreover we have that

$$R(\lambda_j, A_j + B_j) = \left[I - R(\lambda_j, A_j) B_j\right]^{-1} R(\lambda_j, A_j) \qquad (16)$$

and

$$R\left(\lambda_j, (A_j + B_j)^*\right) = \left[I - R\left(\lambda_j, (A_j)^*\right)(B_j)^*\right]^{-1} R\left(\lambda_j, (A_j)^*\right) \qquad (17)$$

for all $\lambda_j \in G$.

Using this lemma and Theorem 3.3, we can exhibit Theorem 3.1.

Proof of Theorem 3.1. We can suppose that $max\left\{\omega^j(T^j), (\lambda_j)_0\right\} < 0$. Else we regard $\left(A_j - \omega^j, D(A_j)\right)$ instead of $\left(A_j, D(A_j)\right)$, where $\omega^j > max\left\{\omega^j(T^j), (\lambda_j)_0\right\}$.

For $x^j \in X$ we define the function $u_x^j : \mathbb{R} \to X$ by

$$u_x^j(1+\varepsilon) := \begin{cases} T^j(1+\varepsilon)x^j, & \varepsilon \geq -1, \\ 0, & \varepsilon < -1. \end{cases}$$

Since $\omega^j(T^j) < 0$, the function u_x^j are in $L^2(\mathbb{R}, X)$ and there is a constant $c \geq 0$ such that

$\left(\int_{-\infty}^{\infty} \left\|\sum_j u_x^j(1+\varepsilon)\right\|^2 d\,(1+\varepsilon)\right)^{1/2} \leq c\left\|\sum_j x^j\right\|$. By Plancherel's Theorem the Fourier transform $\mathcal{F}u_x^j$ of u_x^j are also $L^2(\mathbb{R}, X)$ and $\left\|\sum_j \mathcal{F}u_x^j\right\|_2 = \sqrt{2\pi}\left\|\sum_j u_x^j\right\|_2$. Second hand, we know that

$$\sum_j \left((\mathcal{F}u_x^j)(1+\varepsilon) \right) = \int_{-\infty}^{\infty} \sum_j e^{-i(1+\varepsilon)^2} u_x^j\,(1+\varepsilon) d(1+\varepsilon)$$

$$= \int_0^{\infty} \sum_j e^{-i(1+\varepsilon)^2} T^j\,(1+\varepsilon) x^j d(1+\varepsilon) = R\big(i(1+\varepsilon), A_j\big) x^j$$

for all $(1+\varepsilon) \in \mathbb{R}$. Therefore

$$\left(\int_{-\infty}^{\infty} \left\| \sum_j R\big(i(1+\varepsilon), A_j\big) x^j \right\|^2 d(1+\varepsilon) \right)^{1/2} \leq c\sqrt{2\pi} \left\| \sum_j x^j \right\|. \qquad (18)$$

Using Lemma 5.1, it follows that

$$\left(\int_{-\infty}^{\infty} \left\| \sum_j R\big(i(1+\varepsilon), A_j + B_j\big) x^j \right\|^2 d(1+\varepsilon) \right)^{1/2}$$

$$= \left(\int_{-\infty}^{\infty} \left\| \sum_j \left[I - R\big(i(1+\varepsilon), A_j\big) B_j \right]^{-1} R\big(i(1+\varepsilon), A_j\big) x^j \right\|^2 d(1+\varepsilon) \right)^{1/2}$$

$$\leq \frac{1}{1-M} \left(\int_{-\infty}^{\infty} \left\| \sum_j R\big(i(1+\varepsilon), A_j\big) x^j \right\|^2 d(1+\varepsilon) \right)^{1/2}$$

$$\leq \frac{c\sqrt{2\pi}}{1-M} \left\| \sum_j x^j \right\|$$

for every $x^j \in X$.

We now regard $(A+B)^*$. As before we can exhibit that

$$\left(\int_{-\infty}^{\infty} \left\| \sum_j R\big(i\,\cdot, A_j + B_j\big)^* x^j \right\| \right)^2 \leq \frac{c\sqrt{2\pi}}{1-M} \left\| \sum_j x^j \right\|$$

for every $x^j \in X$. So we can apply Theorem 3.3.

6. Application to Ordinary Differential Operators

Let X be the Hilbert space $L^2(\mathbb{R})$ and $k \in \mathbb{N}$. We regard the sequence of operators $(A_j, D(A_j))$ in X defined by

$$A_j u^j := i\big(u^j\big)^{2k}, \;\; D(A_j) := W^{2k,2}(\mathbb{R}) = \left\{ u^j \in L^2(\mathbb{R}) : \big(u^j\big)^{2k} \in L^2(\mathbb{R}) \right\}. \quad (19)$$

Here $\big(u^j\big)^{2k}$ denotes the $2k$th (distributional) derivative of the function u^j. It is well known that $\big(A_j, D(A_j)\big)$ sequence of generates a C_0-semigroup on .

One can compute that $\mathbb{C}\backslash(i\mathbb{R}) \subseteq \rho(A_j)$ and that for $\lambda_j \in \mathbb{C}\backslash(i\mathbb{R})$ the resolvent of A_j

are given by

$$\sum_j R(\lambda_j, A_j) f_j(x^j) = \frac{i}{2k} \int_{-\infty}^{\infty} \sum_{j=1}^{k} \sum_j \frac{e^{-\mu_j |x^j-(1+\varepsilon)|}}{(-\mu_j)^{2k-1}} f_j(1+\varepsilon) d\,(1+\varepsilon) \quad , x^j \in \mathbb{R},$$

where f_j are a function in $L^2(\mathbb{R})$ and μ_j $(j = 1, \ldots, k)$ are the k solutions of the equation $\lambda_j - i(\mu_j)^{2k} = 0$ with $Re\,\mu_j > 0$.

We now define the sequence of operators $(B_j, D(B_j))$ by

$$B_j f_j := V \cdot f_j^{(l)}, \qquad D(B_j) := \left\{ f_j \in X : V \cdot f_j^{(l)} \in X \right\}, \qquad (20)$$

where V is a potential in $L^2(\mathbb{R})$ and $l \in \mathbb{N}_0$ such that $l < k$.

We want to look at $B_j R(\lambda_j, A_j)$. Take $f_j \in C_c^\infty(\mathbb{R})$, i.e., f_j are in $C^\infty(\mathbb{R})$ and has compact support. For $\lambda_j \in \mathbb{C}/(i\mathbb{R})$ we compute

$$\sum_j B_j R(\lambda_j, A_j) f_j(x^j)$$

$$= \sum_j V(x^j) \cdot \frac{i}{2k} \sum_{j=1}^{k} \sum_j \left(\int_{-\infty}^{x^j} \frac{e^{-\mu_j (x^j-(1+\varepsilon))}}{(-\mu_j)^{2k-l-1}} f_j(1+\varepsilon) d(1+\varepsilon) \right.$$

$$\left. - \int_{-\infty}^{x^j} \frac{e^{-\mu_j (x^j-(1+\varepsilon))}}{(-\mu_j)^{2k-l-1}} f_j(1+\varepsilon) d(1+\varepsilon) \right).$$

Now, if $g^j \in C_c^\infty(\mathbb{R})$ we find

$$\sum_j \left| \langle g^j, B_j R(\lambda_j, A_j) f_j \rangle \right|$$

$$\leq \frac{1}{2k \left| \sum_j \lambda_j \right|^{1-(l+1)/(2k)}} \sum_{j=1}^{k} \sum_j \int_{-\infty}^{\infty} \left| g^i(x^j) \right| \left| V(x^j) \right| \int_{-\infty}^{\infty} e^{-Re\mu_j |x^j-(1+\varepsilon)|} \left| f_j(1+\varepsilon) \right| d(1+\varepsilon) dx^j$$

$$\leq \frac{1}{2k \left| \sum_j \lambda_j \right|^{1-(l+1)/(2k)}} \sum_{j=1}^{k} \sum_j \left(\frac{1}{Re\mu_j} \right)^{1/2} \int_{-\infty}^{\infty} \left| g^j(x^j) \right| \left| V(x^j) \right| dx^j \left\| f_j \right\|_2$$

$$\leq \frac{\|V\|_2}{2 \left| \sum_j \lambda_j \right|^{1-(l+1)/(2k)}} \frac{1}{k} \sum_{j=1}^{k} \sum_j \left(\frac{1}{Re\mu_j} \right)^{1/2} \left\| g^j \right\|_2 \left\| f_j \right\|_2.$$

Since $C_c^\infty(\mathbb{R})$ is dense in L^2 $(\mathbb{R})$, we have shown the estimate

$$\left\| \sum_j B_j R(\lambda_j, A_j) \right\| \leq \frac{\|V\|_2}{2 \left| \sum_j \lambda_j \right|^{1-(l+1)/(2k)}} \frac{1}{k} \sum_{j=1}^{k} \sum_j \left(\frac{1}{Re\mu_j} \right)^{1/2}$$

$$\leq \frac{\|V\|_2}{2\left|\sum_j \lambda_j\right|^{1-(l+1)/(2k)} min\left\{\left(Re\mu_j\right)^{1/2} : j = 1, \cdots, k\right\}}.$$

If $\lambda_j = (1+\varepsilon)e^{i\varphi_j}$ with $\varepsilon > -1$ and $\varphi_j \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, therefore a careful computation yields

$$min\left\{\left(Re\mu_j\right)^{1/2} : j = 1, \ldots, k\right\} = |\lambda|^{1/(4k)}(cos\,\psi_k)^{1/2},$$

where

$$\psi_k = \begin{cases} \frac{\varphi_j}{2k} - \frac{\pi}{4k} + \frac{\pi}{2}, & \text{if } k \text{ is even} \\ \frac{\varphi_j}{2k} + \frac{\pi}{4k} - \frac{\pi}{2}, & \text{if } k \text{ is odd.} \end{cases}$$

Since $|\lambda_j| = Re\,\lambda_j\left(1 + tan^2\,\varphi_j\right)^{1/2} = \frac{Re\,\lambda_j}{cos\varphi_j}$, we have

$$\left|\lambda_j\right|^{1-(l+1)/(2k)} min\left\{\left(Re\mu_j\right)^{1/2} : j = 1, \ldots, k\right\}$$

$$= \left|\lambda_j\right|^{1-(l+1)/(2k)+1/(4k)}(\cos\psi_k)^{1/2}$$

$$= \left(Re\lambda_j\right)^{1-l/(2k)-1/(4k)} \frac{(\cos\psi_k)^{1/2}}{\left(cos\varphi_j\right)^{1-l/(2k)-1/(4k)}}$$

$$= \left(Re\lambda_j\right)^{1-l/(2k)-1/(4k)} \left(\frac{\cos\psi_k}{cos\varphi_j}\right)^{1/2} \left(cos\varphi_j\right)^{-1/2+l/(2k)+1/(4k)}.$$

But $-\frac{1}{2} + \frac{l}{2k} + \frac{1}{4k} = \frac{1}{2k}\left(l - k + \frac{1}{2}\right) \leq 0$, and $\frac{\cos\psi_k}{cos\varphi_j}$ is bounded from below by a constant $c > 0$ for all $\varphi_j \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Therefore

$$\left|\lambda_j\right|^{1-(l+1)/(2k)} min\left\{\left(Re\mu_j\right)^{1/2} : j = 1, \ldots, k\right\} \geq c\left(Re\lambda_j\right)^{1-l/(2k)-1/(4k)}.$$

This exhibits the estimate

$$\left\|\sum_j B_j R(\lambda_j, A_j)\right\| \leq \frac{\|V\|_2}{2c\left(Re\sum_j \lambda_j\right)^{1-l/(2k)-1/(4k)}}. \tag{21}$$

We now can fixate the following proposition.

Proposition 6.1. Let $X = L^2(\mathbb{R})$ and let $(A_j, D(A_j))$ be defined as in (20). If $(B_j, D(B_j))$ are given by

$$B_j f_j := V \cdot f_j^{(l)}, \quad D(B_j) := \left\{f_j \in X : V \cdot f_j^{(l)} \in X\right\},$$

where V is a potential in $L^2(\mathbb{R})$ and $l \in \mathbb{N}_0$ such that $l < k$, therefore the sequence

$(A_j + B_j, D(A_j))$ generates a semigroup on X that are strongly continuous on $(0, \infty)$.

Proof. Since $1 - l/(2k) - 1/(4k) > 0$ by assumption, we obtain from (21) that there is $M < 1$ such that

$$\left\|\sum_j B_j R(\lambda_j, A_j)\right\| \leq M$$

If $Re\lambda_j$ are large enough. It is easy to see that the same is true for $(A_j)^*$ and $(B_j)^*$ instead of A_j and B_j. This yields $\left\|\sum_j R(\lambda_j, A_j) B_j f_j\right\| \leq M \left\|\sum_j f_j\right\|$ for $f_j \in D(B_j)$ and we can apply Theorem 3.1.

Corollary 6.2. Let $X = L^2(\mathbb{R})$ and let $(A_j, D(A_j))$ be defined as in 20. If $V \in L^2(\mathbb{R}) + L^\infty(\mathbb{R})$ and $(B_j, D(B_j))$ are defined as

$$B_j f_j := V \cdot f_j,\ D(B_j) := \{f_j \in X : V \cdot f_j \in X\},$$

Therefore the sequence $(A_j + B_j, D(A_j))$ generates a semigroup on X that are strongly continuous on $(0, \infty)$.

Proof. We split V into an L^2-part and a bounded part. The bounded part can be estimated by the Hille-Yosida theorem. For the L^2 -part, we use again (21) as in the proof of Proposition 6.1.

References

[1] Wolfgang, A. (1987). Resolvent positive operators. Proc. London Math. Soc. [PubMed] [Google Scholar] (3) **54** 321–349. https://doi.org/10.1112/plms/s3-54.2.321.

[2] W. Arendt, C. J. K. Batty, M. Hieber and F. Neubrander (2001). "Vector-Valued Laplace Transforms and Cauchy Problems", Birkh¨auser, Basel,. https://link.springer.com/book/10.1007/978-3-0348-0087-7.

[3] Richard, B. (1972). On the abstract Cauchy problem, J. Functional Analysis **10**, 281–299. https://doi.org/10.1016/0022-1236(72)90027-4.

[4] Adam, B. (1997). On the generation of non-continuous semigroups, Semigroup Forum **54**, 237–252. DOI:10.1007/bf02676605.

[5] Klaus, J, E. and Nagel, R (2000). "One-Parameter Semigroups for Linear Evolution Equations", Graduate Texts in Mathematics, Vol. 194, Springer-Verlag, New York,. https://doi.org/10.1007/s002330010042.

[6] Kreın, S.G. (1971). Linear differential equations in Banach spaces (translated from the Russian by Danskin J. M.), Translations of Mathematical Monographs, Vol. 29, American Mathematical Society, Providence, RI,. https://www.amazon.com/Differential-Equations-Translations-mathematical-monographs/dp/0821815792.

[7] Pazy, A., (1983). "Semigroups of Linear Operators and Applications to Partial Differential Equations", Applied Mathematical Sciences, Vol. 44, Springer- Verlag, New York,. https://link.springer.com/book/10.1007/978-1-4612-5561-1.

[8] Kazuaki, T. (1989). The theory of semigroups with weak singularity and its application to partial differential equations, Tsukuba J. Math. (2) **13**, 513–562. http://dx.doi.org/10.21099/tkbjm/1496161173.

[9] Kaiser, C. and Weis, L. (2003). A Perturbation Theorem for Operator Semigroups in Hilbert Spaces, Verlag New York Inc. Semigroup Forum Vol. **67**, 63–75. DOI: 10.1007/s002330010166.