ANALYTICAL SOLUTION ON COUETTE FLOW

SULIMAN SHEEN¹, ABDELFATAH ABASHER²

¹Deanship of the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharij, Saudi Arabia
EMAIL:sulimanmaleeh@gmail.com
²Mathematics Department, Faculty of Science, Jazan University, Jazan, Saudi Arabia
EMAIL:amoaf84@gmail.com

HNSJ, 2022, 3(10); https://doi.org/10.53796/hnsj3102

Published at 01/10/2022

Abstract

In this paper, we obtain basic flow solutions for stationary viscous flow between two rotating coaxial cylinders by solving the Navier-Stokes's equations in the cylindrical coordinates system $(r, \theta, z)$ for viscous incompressible fluid, simplified the equations and obtained analytically is a Zero – Order Bessel's Function in one variable.

Key Words: viscous flow, rotating, Navier-Stokes's equations, coaxial cylinders, pressure, stationary solution, perturbation equations, couette flow.
حل تحليلي لانسياب كوتي

سليمان شين 1
عبد الفتاح أبشر 2

جامعة الأمير سطام بن عبد العزيز، الخرج، المملكة العربية السعودية
 البريد الإلكتروني: sulimanmaleeh@gmail.com
قسم الرياضيات، كلية العلوم، جامعة جازان، جازان، المملكة العربية السعودية
 البريد الإلكتروني: amoaf84@gmail.com

HNSJ, 2022, 3(10); https://doi.org/10.53796/hnsj3102

المستخلص

في هذه الورقة نحصل على حلول الانسياب الأساسية لانسياب ثابت بين اسطوانتين متحدتين المحور بينهما مائع لا انضغاطي لزج. ذلك بحل معادلات نايفر-استوكس في الاحداثيات الاسطوانية. تم تبسيط المعادلات وحصولاً تحليلياً على دالة بسيطة من الرتبة الصفرية في متغير واحد.

تاريخ النشر: 01/10/2022م
تاريخ القبول: 05/09/2022م
1. INTRODUCTION

The Navier-Stokes's equations for the velocity $u = (u_r, u_\theta, u_z)$ and the pressure $p$ can be written in the form

$$\frac{\partial u_r}{\partial t} + (u \cdot \nabla) u_r - \frac{u^2_\theta}{r} = \frac{\partial}{\partial r} \left( \frac{p}{\rho} \right) + \nu \left( \nabla^2 u_r - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} - \frac{u_r}{r^2} \right)$$  \hspace{1cm} (1.1)

$$\frac{\partial u_\theta}{\partial t} + (u \cdot \nabla) u_\theta - \frac{u_r u_\theta}{r} = \frac{\partial}{\partial \theta} \left( \frac{p}{\rho} \right) + \nu \left( \nabla^2 u_\theta - \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2} \right)$$  \hspace{1cm} (1.2)

$$\frac{\partial u_z}{\partial t} + (u \cdot \nabla) u_z = - \frac{\partial}{\partial z} \left( \frac{p}{\rho} \right) + \nu \nabla^2 u_z$$  \hspace{1cm} (1.3)

in [1]

Where $(u \cdot \nabla) = u_r \frac{\partial}{\partial r} + u_\theta \frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z}$  \hspace{1cm} (1.4)

And $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$  \hspace{1cm} (1.5)

in [2]

The continuity equation in the cylindrical coordinates is given by,

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$  \hspace{1cm} (1.6)

$V(r) = Ar + \frac{B}{r}$  \hspace{1cm} (1.7)

In [3]

These aforementioned equations allow a stationary solution of the form

$$u_r = u_z = 0 \text{ and } u_\theta = r\Omega(r)$$  \hspace{1cm} (1.8)

Thus, Navier-Stokes's equations

Reduced to $\frac{d}{dr} \left( \frac{v}{\rho} \right) = \frac{v^2}{r}$  \hspace{1cm} (1.9)

and $\nu \left( \nabla^2 V - \frac{V}{r^2} \right) = \nu \left( \frac{d}{dr} \left( \frac{d}{dr} + \frac{1}{r} \right) V \right) = 0$ \hspace{1cm} (1.10)

in [3]

2. THE PERTURBATION EQUATIONS AND THE NORMAL MODE

In order to investigate the solutions of the flow system described by equations (1.9) We consider an infinitesimal of the basic flow is given by (1.8) by assuming that the perturbed flow is given by
Assuming that the various perturbations are axisymmetric and independent of θ, and ν = 0 (ideal fluid – water). From (1.1) - (1.3) we gain the following linearized equations as

\[
\begin{align*}
\frac{\partial u_r}{\partial t} - 2 \frac{V}{r} u_{\theta} &= - \frac{\partial \omega}{\partial r} \\
\frac{\partial u_{\theta}}{\partial t} + \left( \frac{dV}{dr} + \frac{V}{r} \right) u_r &= 0
\end{align*}
\]

In [3] and

\[
\frac{\partial u_z}{\partial t} = - \frac{\partial \omega}{\partial z}
\]

where \( \nabla^2 \) is defined by

\[
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}
\]

And the equation of continuity reduces to

\[
\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial^2 u_z}{\partial z^2} = 0
\]

By analyzing the disturbance into normal modes. We assume that the disturbances are of the following form

\[
\begin{align*}
    u_r &= e^{pt} u(r) \cos k z; \quad u_z = e^{pt} \omega(r) \sin k z \\
    u_{\theta} &= e^{pt} v(r) \cos k z \quad \omega = e^{pt} \sigma(r) \cos k z
\end{align*}
\]

in [4]

where k is the wave number of the disturbance in the axial direction, and \( p \) is a constant which can be complex.

Substituting (2.7) in equations (2.2) - (2.6), we get

\[
\begin{align*}
    -pu + 2 \frac{V}{r} v &= \frac{d\omega}{dr} \\
    -pv - (D,V) u &= 0 \\
    p\omega &= k\sigma
\end{align*}
\]

\[
\nabla^2 = \left( \frac{d}{dr} + \frac{1}{r} \right) \frac{d}{dr} - k^2 = D_s D - k^2 = D D_s + \frac{1}{r^2} - k^2
\]
\[ D_s u = -k \omega \]  

(2.12)

In [3]

\[ \text{then} \quad \omega = -\frac{D_s u}{k} \]  

(2.13)

substitute \( \omega \) in equation (2.10), we obtain

\[ D_s u = -\frac{\nu^2}{p} \omega \]  

(2.14)

From equation (2.14), we find

\[ \omega = -\frac{p}{\nu^2} D_s u \]  

(2.15)

Substituting (2.15) in equation (2.8), yields

\[ \frac{p}{k^2} D(D_s u) - pu + 2 \frac{V}{r} v = 0 \]  

(2.16)

By multiplying equation (2.9) by \((2 \frac{V}{r})\), and multiplying equation (2.16) by \(P\)

We obtain,

\[ -2p \frac{V}{r} v - 2 \frac{V}{r} (D_s V) u = 0 \]  

(2.17)

\[ \frac{p^2}{k^2} (DD_s) u - p^2 u + 2p \frac{V}{r} v = 0 \]  

(2.18)

Now, summation equation (2.17) to equation (2.18), we obtain

\[ \frac{p^2}{k^2} (DD_s) u - p^2 u - 2 \frac{V}{r} (D_s V) u = 0 \]  

(2.19)

But, \( D_s = \frac{d}{dr} + \frac{1}{r} \), therefore \((DD_s) = \frac{d^2}{dr^2} + \frac{1}{r^2} \frac{d}{dr} - \frac{1}{r^2} \)

\[ \frac{p^2}{k^2} \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} \right) u - p^2 u - 2 \left( A + \frac{B}{r^2} \right) \left[ \left( \frac{d}{dr} + \frac{1}{r} \right) \left( Ar + \frac{B}{r} \right) \right] u = 0 \]  

(2.21)

angular velocity is

\[ \Omega(r) = A + \frac{B}{r^2}, \quad \Phi(r) = 4A \left( A + \frac{B}{r^2} \right) \]  

(2.22)

in [1]
where $\Phi(r)$ is a real function

By Substituting (2.22) at equation (2.21) We get:

$$
\frac{d^2u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{u}{r^2} - k^2 u + \frac{k^2}{p^2} \Phi u = 0
$$

(2.23)

$$
\frac{d^2u}{dr^2} + \frac{1}{r} \frac{du}{dr} + \left\{ k^2 \left( \frac{\Phi}{p^2} - 1 \right) - \frac{1}{r^2} \right\} u = 0
$$

(2.24)

Equation (2.24) is a Zero – Order Bessel's Function, with the solution

$$
u = b_1 J_0(\gamma r) + b_2 Y_0(\gamma r)
$$

(2.25)

in[5]

where, $\gamma = k \sqrt{\frac{\Phi}{p^2} - 1}$

(2.26)

3. RESULT AND CONCLUSION

(1) Bessel's Function are closely associated with problems processing circular or cylindrical symmetry, because of their close association with cylindrical domains.in[5]

(2) The solutions of Bessel's equation are called cylinder functions. Bessel's Function of the first kind and second kind are special cases of cylinder functions.

(3) $b_1$ and $b_2$ are constants of integration in Equation (2.25), and $J_0$ and $Y_0$ are the Bessel functions of the first and the second kind.

The term $Y_0$ in the solution (2.25) is absent because it is divergent , that means

either $Y_0 \rightarrow 0$ or the constant $b_2 \rightarrow 0$, in this case we have chose $b_2 \rightarrow 0$

We then have the solution

$$u = b_1 J_0(\gamma r)
$$

(2.27)

REFERENCES


rotating Porous cylinders with radial flow”, Department of Mechanical Engineering, Northwestern University, Evanston, Illinois 60208 (Received 28 January 1993; accepted 1 September 1993.