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RESEARCH TITLE

ON REGULARIZING NETS WITH INEQUALITIES AND EQUALITY BETWEEN WEIGHTS

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Abstract

We determine and verify regularizing nets with inequalities and equality between weights**,** we used the deductive method and we found that for the equality of two normal positive forms on a W^*-algebra it is enough that they coincide on a weak*-dense subset. And there are typically many weights which are of little importance in regularizing nets with inqualities and equality between weights.

Key Words: regularizing, nets , inequality, equality, weights.

Introduction

Suppose δ, η be semi-finite, normal weights on a W^*-algebra G, δ faithful and $\eta\ \lambda^{\delta}$-invariant. If $\eta(m^*m) = \delta(m^*m)$ for all m in a weak*-dense subset λ^{δ}-invariant*-subalgebra of $\mathcal{H}_{\delta}$, then $\eta = \delta$. This criterion was further extended in [18] as follows: Let δ, η be as above, and p a positive element of the centralizer of δ. If $\eta(m^*m) = \delta\left(\sqrt{p}m^*m\sqrt{p}\right)$ for m in a weak*-dense subset λ^{δ}-invariant*-subalgebra of $\mathcal{H}_{\delta}$ then $\eta = \delta\left(\sqrt{p}\,.\sqrt{p}\right)$.

Regularizing nets are useful in the modular theory of faithful, semi-finite, normal weight. Suppose G be W^*-algebra, and δ a faithful, semi-finite, normal weight on G. We call regularizing net for δ any net $\left(h_{\rho}\right)_{\rho}$ in Ω_{δ} such that

(i) $\sup\limits_{(1-\epsilon)\underset{\rho}{\in} Q} \left\|\lambda^{\delta}_{(1-\epsilon)}\left(h_{\rho}\right)\right\| < +\infty$ and $\sup\limits_{(1-\epsilon)\underset{\rho}{\in} Q} \left\|\lambda^{\delta}_{(1-\epsilon)}\left(h_{\rho}\right)_{\delta}\right\| < +\infty$ for each compact $Q \subset \mathbb{C}$;

(ii) $\lambda^{\delta}_{(1-\epsilon)}\left(h_{\rho}\right) \overset{\rho}{\rightarrow} H_G$ in the T^*-topology for all $(1-\epsilon) \subset \mathbb{C}$.

In the modular theory of faithful, semi-finite, normal weights the regularizing nets are useful. Ususlly they are constructed starting with a bounded net $\left(m_{\rho}\right)_{\rho}$ in Ω_{δ} such that $m_{\rho} \overset{\rho}{\rightarrow} H_G$ in the T^*-topology and then letting it ‘‘mollified’’, for modle, by the mollifier $e^{-(1+\epsilon)^2}$, that is passing to the net $\left(h_{\rho}\right)_{\rho}$ then

$$h_{\rho} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(1+\epsilon)^2}\, \lambda^{\delta}_{(1-\epsilon)}\left(m_{\rho}\right) d(1+\epsilon). \qquad (1)$$

The verification of (i) is straightforward, more troublesome is to verify the inclusion $h_{\rho} \in \Omega_{\delta}$ and the convergence (ii).

Concerning the verification of (ii), if the net $\left(m_{\rho}\right)_{\rho}$ would be increasing, we could proceed as in the proof of [13] by using Dini’s theorem. But there are situations in which we cannot restrict us to the case of increasing $\left(m_{\rho}\right)_{\rho}$. For example, it is not clear whether every T^*-dense, λ^{δ}-invariant (not necessarily hereditary)*-subalgebra of G contains some increasing net $\left(m_{\rho}\right)_{\rho}$ with $m_{\rho} \overset{\rho}{\rightarrow} H_G$ in the T^*-topology as used in the proof of [13].

By the other side, if the net $\left(m_{\rho}\right)_{\rho}$would be a sequence, we can use the dominated convergence theorem of Lebesgue, similarly as, for example, in the proof of [14],Theorem 2.16. But again, unless G is countably decomposable (and so its unit ball T^*-metrizable), the unit ball of not every T^*-dense*-subalgebra of G contains a sequence T^*-convergent to H_G. Here we notice that Lebesgue theorem of convergence is very useful in this case also we can cover the other case of non countable nets $\left(m_{\rho}\right)_{\rho}$ to determine and verify (ii) directly, using advantage of the particularites of the situation. Here we will prove that, starting with a bounded net $\left(m_\rho\right)_\rho$ even in ϑ_δ, equation (1) furnishes a regularizing net $\left(h_\rho\right)_\rho$[19].

The next lemma is [2] equation (2.27) it is also another type of the modular theory of faithful semi-finite, normal weights concerns some facts.

Lemma 1. Let δ be a faithful, semi-finite, normal weight on a W^*-algebra G. If $m \in \vartheta_\delta$ and $g \in E^1(\mathbb{R})$, then

$$\int_{-\infty}^{+\infty} g(1+\epsilon)\,\lambda^{\delta}_{(1+\epsilon)}(m)d(1+\epsilon) \in \vartheta_\delta \left(\int_{-\infty}^{+\infty} g(1+\epsilon)\,\lambda^{\delta}_{(1+\epsilon)}(m)d(1+\epsilon)\right)_\delta$$

$$= \int_{-\infty}^{+\infty} g(1+\epsilon)\,\Delta_\delta^{i(1+\epsilon)} d(1+\epsilon).$$

Let δ be a faithful, semi-finite, normal weight on a W^*-algebra G and $(1-\epsilon) \in \mathbb{C}$. We define the linear operator $\lambda^\delta_{(1-\epsilon)}: G \supset U\left(\lambda^\delta_{(1-\epsilon)}\right) \ni m \longmapsto \lambda^\delta_{(1-\epsilon)}(m) \in G$ as follows: the pair $\left(m, \lambda^\delta_{(1-\epsilon)}(m)\right)$ belongs to its graph whenever the map $\mathbb{R} \ni (1+\epsilon) \longmapsto \lambda^\delta_{(1+\epsilon)}(m) \in G$ has a ω-continuous extension on the closed strip $\{\Omega \in \mathbb{C}; 0 \le |\mathrm{lm}\,\Omega| \le |\mathrm{lm}(1-\epsilon)|, (\mathrm{lm}\,\Omega)(\mathrm{lm}(1-\epsilon)) \le 0\}$,

Analytic in the interior and taking the value $\lambda^\delta_{(1-\epsilon)}(m)$ at $(1-\epsilon)$. It is easily seen (see e.g. [17],Theorem 1.6) that, for each $(1-\epsilon) \in \mathbb{C}$,

$$\mathcal{P}\left(\lambda^\delta_{(1-\epsilon)}\right)^* = \mathcal{P}\left(\lambda^\delta_{\overline{(1-\epsilon)}}\right) \text{ and } \lambda^\delta_{\overline{(1-\epsilon)}}(m^*) = \lambda^\delta_{(1-\epsilon)}(m)^* \qquad (2)$$

for every $m \in \mathcal{P}\left(\lambda^\delta_{(1-\epsilon)}\right)$.

We recall that $m \in G$ belongs to $\mathcal{P}\left(\lambda^\delta_{(1-\epsilon)}\right)$ if and only if the operator $\Delta_\delta^{i(1-\epsilon)}\pi_\delta(m)\Delta_\delta^{-i(1-\epsilon)}$ is defined and bounded on a core of $\Delta_\delta^{-i(1-\epsilon)}$, in which case $\mathcal{P}\left(\Delta_\delta^{i(1-\epsilon)}\pi_\delta(m)\Delta_\delta^{-i(1-\epsilon)}\right) = \mathcal{P}\left(\Delta_\delta^{-i(1-\epsilon)}\right)$ and $\Delta_\delta^{i(1-\epsilon)}\pi_\delta(m)\Delta_\delta^{-i(1-\epsilon)} = \pi_\delta\left(\lambda^\delta_{(1-\epsilon)}(m)\right)$ that is

$$\pi_\delta(m)\Delta_\delta^{-i(1-\epsilon)} \subset \Delta_\delta^{-i(1-\epsilon)}\pi_\delta\left(\lambda^\delta_{(1-\epsilon)}(m)\right) \qquad (3)$$

(see [3],Theorem 6.2 or [2], Theorem 2.3).

Here we determine and verify the form of an element of $\vartheta_\delta \Longrightarrow \vartheta_\delta^*$ hence to γ_δ:

Lemma 2. Let δ be a faithful, semi-finite, normal weight on a W^*-algebra G, $m \in \mathcal{P}\left(\lambda^\delta_{\frac{-i}{2}}\right)$ and $\lambda^\delta_{\frac{-i}{2}}(m) \in \vartheta_\delta \Longrightarrow m \in \vartheta_\delta^*$ and $\lambda^\delta_{\frac{-i}{2}}(m)_\delta = \Delta_\delta^{\frac{1}{2}} m_\delta$, that is $m \in \gamma_\delta$ and $T_\delta m_\delta = X_\delta \lambda^\delta_{\frac{-i}{2}}(m)_\delta$.

Proof. Let $n \in \mathcal{H}_\delta$ be arbitrary. Then

$$\pi_\delta(m^*)X_\delta n_\delta = \pi_\delta(m^*)X_\delta(T_\delta(n^*)_\delta) = \pi_\delta(m^*)\Delta_\delta^{\frac{1}{2}}(n^*)_\delta. \quad (4)$$

Application of (2) with $\left(\epsilon = \frac{i}{2} - 1\right)$ yields $m^* \in \mathcal{P}\left(\lambda^{\delta}_{\frac{i}{2}}\right)$ and $\lambda^{\delta}_{\frac{i}{2}}(m^*) = \lambda^{\delta}_{\frac{-i}{2}}(m)^*$, so, applying (3) to m^* and $\left(\epsilon = -\frac{i}{2} + 1\right)$, we deduce

$$\pi_{\delta}(m^*)\Delta_{\delta}^{\frac{1}{2}} \subset \Delta_{\delta}^{\frac{1}{2}}\pi_{\delta}\left(\lambda^{\delta}_{\frac{i}{2}}(m^*)\right) = \Delta_{\delta}^{\frac{1}{2}}\pi_{\delta}\left(\lambda^{\delta}_{\frac{-i}{2}}(m)^*\right). \quad (5)$$

By (4) and (5) we conclude:

$$\pi_{\delta}(m^*)X_{\delta}n_{\delta} = \Delta_{\delta}^{\frac{1}{2}}\pi_{\delta}\left(\lambda^{\delta}_{\frac{-i}{2}}(m)^*\right)(n^*)_{\delta} = \Delta_{\delta}^{\frac{1}{2}}\left(\lambda^{\delta}_{\frac{-i}{2}}(m)^* n^*\right)_{\delta} =$$

$$X_{\delta}T_{\delta}\left(\left(n\lambda^{\delta}_{\frac{-i}{2}}(m)\right)^*\right)_{\delta} = X_{\delta}\left(n\lambda^{\delta}_{\frac{-i}{2}}(m)\right)_{\delta} = X_{\delta}\pi_{\delta}(n)\lambda^{\delta}_{\frac{-i}{2}}(m)_{\delta}.$$

By the aboves $\|\pi_{\delta}(m^*)X_{\delta}n_{\delta}\| \leq \left\|\lambda^{\delta}_{\frac{-i}{2}}(m)_{\delta}\right\| . \|n\|, \quad n \in \mathcal{H}_{\delta},$

applying [2], Lemma 2.6 (1) to deduce that $m^* \in \vartheta_{\delta} \Leftrightarrow m \in \vartheta_{\delta}^*$ [19].

Taking into account that $m \in \gamma_{\delta}$ and $n \in \mathcal{H}_{\delta} \subset \gamma_{\delta}$, and using [2], (5), as well as the above (3) with $\left(\epsilon = \frac{i}{2} - 1\right)$, we deduce:

$$\pi_{\delta}(n)X_{\delta}\lambda^{\delta}_{\frac{-i}{2}}(m)_{\delta} = X_{\delta}\pi_{\delta}\left(\lambda^{\delta}_{\frac{-i}{2}}(m)\right)X_{\delta}n_{\delta} = X_{\delta}\pi_{\delta}\left(\lambda^{\delta}_{\frac{-i}{2}}(m)\right)X_{\delta}T_{\delta}(n^*)_{\delta} =$$

$$X_{\delta}\pi_{\delta}\left(\lambda^{\delta}_{\frac{-i}{2}}(m)\right)\Delta_{\delta}^{\frac{1}{2}}(n^*)_{\delta} = X_{\delta}\Delta_{\delta}^{\frac{1}{2}}\pi_{\delta}(m)(n^*)_{\delta} = T_{\delta}(mn^*)_{\delta} = (nm^*)_{\delta} =$$

$$\pi_{\delta}(n)(m^*)_{\delta} = \pi_{\delta}(n)T_{\delta}m_{\delta} = \pi_{\delta}(n)X_{\delta}\Delta_{\delta}^{\frac{1}{2}}m_{\delta}.$$

Since $\pi_{\delta}(\mathcal{H}_{\delta})$ is ω-dense in G, it follows the equality $\lambda^{\delta}_{\frac{-i}{2}}(m)_{\delta} = \Delta_{\delta}^{\frac{1}{2}}m_{\delta}$.

The above two lemmas can be used to produce elements of the Tomita algebra Ω_{δ} by ‘‘regularizing’’ elements of ϑ_{δ} (not only elements of γ_{δ}, as customary : (see in [15], the comments after the proof of Theorem 10.20 on page 347) :

Lemma 3. Let δ be a faithful, semi-finite, normal weight on a W^*-algebra G.
For each $m \in G$.

$$h = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(1+\epsilon)^2}\,\lambda^{\delta}_{(1+\epsilon)}(m)d(1+\epsilon)$$

belongs to G^{δ}_{∞} and

$$\lambda^{\delta}_{(1-\epsilon)}(h) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(2\epsilon)^2}\,\lambda^{\delta}_{(1+\epsilon)}(m)d(1+\epsilon), \qquad (6)$$

$(1-\epsilon) \in \mathbb{C}$.

We assume that $m \in \vartheta_{\delta}$, we get $h \in \Omega_{\delta}$.

Proof. If

$$\mathbb{R} \ni (1-2\epsilon) \mapsto \lambda^{\delta}_{(1-2\epsilon)}(h) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(1+\epsilon)^2}\,\lambda^{\delta}_{(2-\epsilon)}(m)d(1+\epsilon)$$
$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{\epsilon^2}\,\lambda^{\delta}_{(1+\epsilon)}(m)d(1+\epsilon)$$

allows the entire extension

$$\mathbb{C} \ni (1-\epsilon) \mapsto \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(2\epsilon)^2}\,\lambda^{\delta}_{(1+\epsilon)}(m)d(1+\epsilon),$$

we have $h \in G^{\delta}_{\infty}$ and (6) holds true.

By assuming if $m \in \vartheta_{\delta}$, we have $\lambda^{\delta}_{(1-\epsilon)}(h) \in \vartheta_{\delta},\ (1-\epsilon) \in \mathbb{C}$.

Using (6) it is easy to see that

$$\lambda^{\delta}_{(1+\epsilon)}\left(\lambda^{\delta}_{(1-\epsilon)}(h)\right) = \lambda^{\delta}_{0}(h),\ \ (1-\epsilon) \in \mathbb{C}, (1+\epsilon) \in \mathbb{R},$$

so

$$\lambda^{\delta}_{(1-\epsilon)}(h) \in G^{\delta}_{\infty} \text{ and } \lambda^{\delta}_{\Omega}\left(\lambda^{\delta}_{(1-\epsilon)}(h)\right) = \lambda^{\delta}_{(1-\epsilon)+\Omega}(h), \qquad (1-\epsilon), \Omega \in \mathbb{C}. \qquad (7)$$

For each $(1-\epsilon) \in \mathbb{C}$, applying Lemma 1 with $g(1+\epsilon) = \frac{1}{\sqrt{\pi}}e^{-(2\epsilon)^2}$, we deduce that $\lambda^{\delta}_{(1-\epsilon)}(h) \in \vartheta_{\delta}$. Since $(1-\epsilon) \in \mathbb{C}$ is here arbitrary, also $\lambda^{\delta}_{(1-\epsilon)-\frac{i}{2}}(h) \in \vartheta_{\delta}$ holds true. But by (7) we get $\lambda^{\delta}_{\frac{-i}{2}}\left(\lambda^{\delta}_{(1-\epsilon)}(h)\right) = \lambda^{\delta}_{(1-\epsilon)-\frac{i}{2}}(h)$, so $\lambda^{\delta}_{\frac{-i}{2}}\left(\lambda^{\delta}_{(1-\epsilon)}(h)\right) \in \vartheta_{\delta}$. Applying now Lemma 2, we conclude that $\lambda^{\delta}_{(1-\epsilon)}(h)$ belongs also to ϑ^{*}_{δ}, hence $\lambda^{\delta}_{(1-\epsilon)}(h) \in \gamma_{\delta}$.

By using the integrals of equation (6) and Lemma 4 we can prove the dominated convergence theorem for integrals and nets.

Lemma 4. Take δ as a faithful, semi-finite, normal weight on a W^{*}-algebra G. and $(m_{\rho})_{\rho}$ a net in the closed unit ball of G such that $m_{\rho} \overset{\rho}{\to} H_G$ in the T^{*}-topology. Let the net $(h_{\rho})_{\rho}$ be defined by the equation

$$h_{\rho} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(1+\epsilon)^2}\,\lambda^{\delta}_{(1+\epsilon)}(m_{\rho})d(1+\epsilon).$$

Then

(i) $h_{\rho} \in G^{\delta}_{\infty}$ for all ρ;

(ii) $\left\|\lambda^{\delta}_{(1-\epsilon)}(h_{\rho})\right\| \leq e^{(\mathrm{Im}(1-\epsilon))^2}$ for all ρ and $(1-\epsilon) \in \mathbb{C}$;

(iii) $\lambda^{\delta}_{(1-\epsilon)}(h_{\rho}) \overset{\rho}{\to} H_G$ in the T^{*}-topology for all $(1-\epsilon) \in \mathbb{C}$.

Proof. (i) is immediate consequence of Lemma 3.

For (ii), let ρ and $(1-\epsilon) \in \mathbb{C}$ be arbitrary. By Lemma 3 we have

$$\lambda^{\delta}_{(1-\epsilon)}(h_{\rho}) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(2\epsilon)^2}\,\lambda^{\delta}_{(1+\epsilon)}(m_{\rho})d(1+\epsilon). \qquad (8)$$

Since $\left\|\lambda^{\delta}_{(1-\epsilon)}(m_{\rho})\right\| = \left\|m_{\rho}\right\| \leq 1$ for all $(1+\epsilon) \in \mathbb{R}$, it follows

$\left\|\lambda^{\delta}_{(1-\epsilon)}\left(h_{\rho}\right)\right\| \le \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(2\epsilon)^2} \,|d(1+\epsilon) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(1+\epsilon-\mathrm{Re}z)^2+(\mathrm{lm}(1-\epsilon))^2}\, d(1+\epsilon) = e^{(\mathrm{lm}(1-\epsilon))^2}$.

The more involved issue is (iii). For fixed $(1-\epsilon)\in\mathbb{C}$, we have to show that

$$\lambda^{\delta}_{(1-\epsilon)}\left(h_{\rho}\right) - H_G = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(2\epsilon)^2}\, \lambda^{\delta}_{(1+\epsilon)}\left(m_{\rho}-H_G\right) d(1+\epsilon) \overset{\rho}{\rightarrow} 0$$

in the T^*-topology. Since the T^*-topology is definded by the semi-norms $u_{\eta}\colon G \ni m \longmapsto \eta\sqrt{m^*m} + \eta\sqrt{mm^*}$, η a normal positive form on G, then

$$u_{\eta}\left(\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(2\epsilon)^2}\, \lambda^{\delta}_{(1+\epsilon)}\left(m_{\rho}-H_G\right) d(1+\epsilon)\right) \overset{\rho}{\rightarrow} 0$$

for every a normal positive form η on G.

For let η be any a normal positive form η on G. Since, according to [19], equation (3), $u_{\eta}\left(\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(2\epsilon)^2}\, \lambda^{\delta}_{(1+\epsilon)}\left(m_{\rho}-H_G\right) d(1+\epsilon)\right) \le \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} u_{\eta}\left(e^{-(2\epsilon)^2}\lambda^{\delta}_{(1+\epsilon)}\left(m_{\rho}-H_G\right)\right) d(1+\epsilon) = \int_{-\infty}^{+\infty}\left|e^{-(2\epsilon)^2}\right| u_{\eta}\left(\lambda^{\delta}_{(1+\epsilon)}\left(m_{\rho}-H_G\right)\right) d(1+\epsilon)$, if we prove the convergence the proof will be complete.

$\int_{-\infty}^{+\infty}\left|e^{-(2\epsilon)^2}\right| u_{\eta}\left(\lambda^{\delta}_{(1+\epsilon)}\left(m_{\rho}-H_G\right)\right) d(1+\epsilon) \overset{\rho}{\rightarrow} 0,$

that is consequence of

$\int_{-\infty}^{+\infty} e^{-(1+\epsilon-\mathrm{Re}z)^2+(\mathrm{lm}(1-\epsilon))^2}\left(\eta\circ\lambda^{\delta}_{(1+\epsilon)}\right)\left(\left(m_{\rho}-H_G\right)^*\left(m_{\rho}-H_G\right)+\left(m_{\rho}-H_G\right)\left(m_{\rho}-H_G\right)^*\right)^{\frac{1}{2}} d(1+\epsilon) \overset{\rho}{\rightarrow} 0$.   (9)

because $\left|e^{-(2\epsilon)^2}\right| = e^{-(1+\epsilon-\mathrm{Re}z)^2+(\mathrm{lm}(1-\epsilon))^2}$ and

$u_{\eta}\left(\lambda^{\delta}_{(1+\epsilon)}\left(m_{\rho}-H_G\right)\right) = \left(\eta\circ\lambda^{\delta}_{(1+\epsilon)}\right)\left(\left(m_{\rho}-H_G\right)^*\left(m_{\rho}-H_G\right)\right)^{\frac{1}{2}} + \left(\eta\circ\lambda^{\delta}_{(1+\epsilon)}\right)\left(\left(m_{\rho}-H_G\right)\left(m_{\rho}-H_G\right)^*\right)^{\frac{1}{2}} \le \sqrt{2}\left(\eta\circ\lambda^{\delta}_{(1+\epsilon)}\right)\left(\left(m_{\rho}-H_G\right)^*\left(m_{\rho}-H_G\right)+\left(m_{\rho}-H_G\right)\left(m_{\rho}-H_G\right)^*\right)^{\frac{1}{2}}$.

The proof will be complete by using verifying (9) [19].

Since $m_{\rho}\overset{\rho}{\rightarrow}H_G$ in the T^*-topology and $\left\|m_{\rho}\right\|\le 1$ for all ρ , we have that

$$\left(\left(m_{\rho}-H_G\right)^*\left(m_{\rho}-H_G\right)+\left(m_{\rho}-H_G\right)\left(m_{\rho}-H_G\right)^*\right)_{\rho}$$

is a bounded net, convergent to 0 in the T^*-topology. According to a theorem due to Akemann (see [1], Theorem II.7 or [16], Corollary 8.17), on bounded subsets of G the T^*-topology coincides with the Mackey topology τ_{ω} associated to the the ω-topology, that is with the topology of the uniform convergence on the weakly compact absolutely convex subsets of the predual G_*. Since, by the classical Krein-Šmulian theorem (see e.g. [9], Theorem V.6.4), the closed absolutely convex hull of every weakly compact set in Banach space is still weakly compact, τ_{ω} is actually the topology of the uniform convergence on the weakly compact subsets of G_*.Therefore

$$\sup_{\theta\in Q}\left|\theta\left(\left(m_\rho-H_G\right)^*\left(m_\rho-H_G\right)+\left(m_\rho-H_G\right)\left(m_\rho-H_G\right)^*\right)\right| \xrightarrow{\rho} 0 \ (10)$$

for every weakly compact $Q\subset G_*$.

Now let $\epsilon>0$ be arbitrary. Choose some $(1+\epsilon)_0>0$, then

$$\int_{|1+\epsilon|>(1+\epsilon)_0} e^{-(1+\epsilon-\mathrm{Rez})^2}\, d(1+\epsilon)\le\frac{\epsilon}{4\sqrt{2\|\vartheta\|}}. \quad (11)$$

Since $Q_{(1+\epsilon)_0}=\left\{\eta\circ\lambda^{\delta}_{(1+\epsilon)};|1+\epsilon|\le(1+\epsilon)_0\right\}$ is a weakly compact subset of G_*, (10) holds true with $Q=Q_{(1+\epsilon)_0}$ .Then there exists some ρ_0 such that

$$\sup_{|1+\epsilon|>(1+\epsilon)_0}\left|\left(\eta\circ\lambda^{\delta}_{(1+\epsilon)}\right)\left(\left(m_\rho-H_G\right)^*\left(m_\rho-H_G\right)+\left(m_\rho-H_G\right)\left(m_\rho-H_G\right)^*\right)\right| \le\frac{\epsilon}{2\sqrt{\pi}} \qquad (12)$$

for all $\rho\ge\rho_0$. (11) implies

$\int_{|1+\epsilon|>(1+\epsilon)_0} e^{-(1+\epsilon-\mathrm{Rez})^2}\left(\eta\circ\lambda^{\delta}_{(1+\epsilon)}\right)\left(\left(m_\rho-H_G\right)^*\left(m_\rho-H_G\right)+\left(m_\rho-H_G\right)\left(m_\rho-H_G\right)^*\right)^{\frac{1}{2}} d(1+\epsilon)\le\int_{|1+\epsilon|>(1+\epsilon)_0} e^{-(1+\epsilon-\mathrm{Rez})^2}(8\|\eta\|)^{\frac{1}{2}} d(1+\epsilon)\le\frac{\epsilon}{4\sqrt{2\|\eta\|}}(8\|\eta\|)^{\frac{1}{2}}=\frac{\epsilon}{2}$

,

while using (12) we deduce for every $\rho\ge\rho_0$:

$\int_{|1+\epsilon|\le(1+\epsilon)_0} e^{-(1+\epsilon-\mathrm{Rez})^2}\left(\eta\circ\lambda^{\delta}_{(1+\epsilon)}\right)\left(\left(m_\rho-H_G\right)^*\left(m_\rho-H_G\right)+\left(m_\rho-H_G\right)\left(m_\rho-H_G\right)^*\right)^{\frac{1}{2}} d(1+\epsilon)\le\int_{|1+\epsilon|\le(1+\epsilon)_0} e^{-(1+\epsilon-\mathrm{Rez})^2}\, d(1+\epsilon)\le\frac{\epsilon}{2\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(1+\epsilon-\mathrm{Rez})^2} d(1+\epsilon)=\frac{\epsilon}{2\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(1+\epsilon)^2} d(1+\epsilon)=\frac{\epsilon}{2}$.

Consequently for every $\rho\ge\rho_0$,

$\int_{-\infty}^{+\infty} e^{-(1+\epsilon-\mathrm{Rez})^2}\left(\eta\circ\lambda^{\delta}_{(1+\epsilon)}\right)\left(\left(m_\rho-H_G\right)^*\left(m_\rho-H_G\right)+\left(m_\rho-H_G\right)\left(m_\rho-H_G\right)^*\right)^{\frac{1}{2}} d(1+\epsilon)=\int_{|1+\epsilon|>(1+\epsilon)_0}\ldots+\int_{|1+\epsilon|<(1+\epsilon)_0}\ldots\ \le\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.$

Theorem 5. Let δ be a faithful, semi-finite, normal weight on a W^*-algebra G. and $\left(m_\rho\right)_\rho$ a net in the closed unit ball of G such that $m_\rho\xrightarrow{\rho}H_G$ in the T^*-topology. Let the net $\left(h_\rho\right)_\rho$ we define it by the equation

$$h_\rho=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(1+\epsilon)^2}\,\lambda^{\delta}_{(1+\epsilon)}\left(m_\rho\right) d(1+\epsilon).$$

Then

(i) $h_\rho\in G^{\delta}_{\infty}$ for all ρ;

(ii) $\left\|\lambda^{\delta}_{(1-\epsilon)}\left(h_\rho\right)\right\|\le e^{(\mathrm{Im}(1-\epsilon))^2}$ for all ρ and $(1-\epsilon)\in\mathbb{C}$;

(iii) $\lambda^{\delta}_{(1-\epsilon)}(h_\rho) \overset{\rho}{\rightarrow} H_G$ in the T^*-topology for all $(1-\epsilon) \in \mathbb{C}$.

Futhermore, if $m_\rho \in \vartheta_\delta$ for all ρ , hence h_ρ belongs to Ω_ρ for every ρ and therefore $(h_\rho)_\rho$ is a regularizing net for δ.

For determing and verifying criteria for inequalities and equalities between weights we use the generalization of [18], Lemma 2.1.

By recalling that p^*-subalgebra $\mathcal{H}$ of a W^*-algebra G is called facial subalgebra or hereditary subalgebra whenever $\mathcal{H} \cap G^+$ is a face, that is a convex cone satisfying $G^+ \ni q \le p \in \mathcal{H} \cap G^+ \Longrightarrow q \in \mathcal{H} \cap G^+$. and $\mathcal{H}$ is the linear span of it (see e.g. [15], Section 3.21).

Theorem 6. Let G be a W^*-algebra, δ a faithful, semi-finite, normal weight on $G, p \in (G^\delta)^+$ and η a normal weight on G. Assume that there exists a ω-dense, λ^δ-invariant*- subalgebra $\mathcal{H}$ of $\mathcal{H}_{\delta_p}$ such that $\eta(m^*m) = \delta_p(m^*m), \quad m \in \mathcal{H}$.Then

$$\eta \le \delta_p \,. \qquad (13)$$

Then, there exists a λ^δ- invariant, hereditary*-subalgebra $\mathcal{H}_0$ of $\mathcal{H}_{\delta_p}$ such that $\mathcal{H} \cap G^+ \subset \mathcal{H}_0 \cap G^+, \eta(q) \le \delta_p(q), \quad q \in \mathcal{H}_0 \cap G^+$.

The difference between the above Theorem 6 and [18], Lemma 2.1 consists in the fact that in [18], Lemma 2.1 is additionally assumed that

(i) η is semi-finite and λ^δ- invariant and

(ii) $\mathcal{H}$ is contained already in $\mathcal{H}_\delta$ (which of course, according to [13], Theorem 3.6, is a subset of $\mathcal{H}_{\delta_p}$).

However the proof of [18], Lemma 2.1 does not use assumption (i) and, by the other side, we can adapt it to work with the assumption $\mathcal{H} \subset \mathcal{H}_{\delta_p}$

Proof. Let $m \in \mathcal{H} \subset \mathcal{H}_{\delta_p}$ be arbitrary. Since $\eta(m^*m) = \delta_p(m^*m) < +\infty$, we have $m \in \vartheta_\eta \cap \vartheta_{\delta_p}$ and therefore $\eta(m^*.\ m)$ and $\delta_p(m^*.\ m)$ are normal positive forms on G. We notice that $\eta(m^*n^*nm) = \delta_p(m^*n^*nm)$, and $\mathcal{H}$ is ω-dense in G, we deduce that

$$\eta(m^*.\ m) = \delta_p(m^*.\ m) \qquad (14)$$

By the density theorem of Kaplansky there exists a net $(p_\rho)_\rho$ in $\mathcal{H}$ such that $0 \le p_\rho \le H_G$ for all ρ and $p_\rho \xrightarrow{(1-2\epsilon)^*} H_G$. Set, for each ρ,

$$h_\rho = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(1+\epsilon)^2} \lambda^{\delta}_{(1+\epsilon)}(p_\rho) d(1+\epsilon) \ \in \ G^+. \qquad (15)$$

Clearly, $0 \le h_\rho \le H_G$ for all ρ. According to Lemma 4, $h_\rho \in G^\delta_\infty$ for all ρ and

$$\lambda^{\delta}_{(1-\epsilon)}(h_\rho) \overset{\rho}{\rightarrow} H_G \qquad (16)$$

in the T^*-topology for all $(1-\epsilon) \in \mathbb{C}$. Since $\sqrt{p} \in G^\delta$, also

$$h_\rho\sqrt{p} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(1+\epsilon)^2}\,\lambda_{(1+\epsilon)}^{\delta}(p_\rho)\sqrt{p}d(1+\epsilon)$$
$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(1+\epsilon)^2}\,\lambda_{(1+\epsilon)}^{\delta}(p_\rho\sqrt{p})d(1+\epsilon) \quad (17)$$

belongs to G_∞^δ for each ρ . Furthermore, $p_\rho \in \mathcal{H} \subset \mathcal{H}_{\delta_a}$ yields

$$\delta(p_\rho\sqrt{p})^*(p_\rho\sqrt{p}) = \delta_p(p_\rho^2) < +\infty.$$

hence $p_\rho\sqrt{p} \in \vartheta_\delta$. We apply Lemma 3 and (17) we deduce that $p_\rho\sqrt{p} \in \Omega_\delta$ for all ρ.

Let $n \in G$ and ρ be arbitrary. Since $p_\rho \in \mathcal{H}$ and $\mathcal{H}$ is λ^δ–invariant, application of (14) yields for every $(1+\epsilon)$, $(1-2\epsilon) \in \mathbb{R}$ and $t = 0, 1, 2, 3$:

$$\eta\left(\left(\lambda_{(1+\epsilon)}^{\delta}(p_\rho) + i^t\lambda_{(1-2\epsilon)}^{\delta}(p_\rho)\right)^* n^*n\left(\lambda_{(1+\epsilon)}^{\delta}(p_\rho) + i^t\lambda_{(1-2\epsilon)}^{\delta}(p_\rho)\right)\right)$$
$$= \delta_p\left(\left(\lambda_{(1+\epsilon)}^{\delta}(p_\rho) + i^t\lambda_{(1-2\epsilon)}^{\delta}(p_\rho)\right)^* n^*n\left(\lambda_{(1+\epsilon)}^{\delta}(p_\rho)\right.\right.$$
$$\left.\left. + i^t\lambda_{(1-2\epsilon)}^{\delta}(p_\rho)\right)\right).$$

We apply [19], equation (1.2) with

$$V(1+\epsilon, 1-2\epsilon)$$
$$= \frac{1}{\pi}e^{-(1+\epsilon)^2-(1-2\epsilon)^2}\left(\lambda_{(1+\epsilon)}^{\delta}(p_\rho) + i^t\lambda_{(1-2\epsilon)}^{\delta}(p_\rho)\right)^* n^*n\left(\lambda_{(1+\epsilon)}^{\delta}(p_\rho)\right.$$
$$\left. + i^t\lambda_{(1-2\epsilon)}^{\delta}(p_\rho)\right),$$

it follows for $t = 0, 1, 2, 3$:

$$\eta\left(\tfrac{1}{\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(1+\epsilon)^2-(1-2\epsilon)^2}\left(\lambda_{(1+\epsilon)}^{\delta}(p_\rho) + i^t\lambda_{(1-2\epsilon)}^{\delta}(p_\rho)\right)^* n^*n\left(\lambda_{(1+\epsilon)}^{\delta}(p_\rho) + i^t\lambda_{(1-2\epsilon)}^{\delta}(p_\rho)\right) d(1+\epsilon)\, d(1-2\epsilon)\right) = \delta_p\left(\tfrac{1}{\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(1+\epsilon)^2-(1-2\epsilon)^2}\left(\lambda_{(1+\epsilon)}^{\delta}(p_\rho) + i^t\lambda_{(1-2\epsilon)}^{\delta}(p_\rho)\right)^* n^*n\left(\lambda_{(1+\epsilon)}^{\delta}(p_\rho) + i^t\lambda_{(1-2\epsilon)}^{\delta}(p_\rho)\right) d(1+\epsilon)\, d(1-2\epsilon)\right).$$

Since, by (15),

$$h_\rho n^*nh_\rho = \frac{1}{\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(1+\epsilon)^2-(1-2\epsilon)^2}\lambda_{(1-2\epsilon)}^{\delta}(p_\rho)n^*n\lambda_{(1+\epsilon)}^{\delta}(p_\rho)d(1+\epsilon)\, d(1$$
$$- 2\epsilon)$$
$$= \frac{1}{4}\sum_{t=0}^{3}\frac{i^t}{\pi}\frac{1}{\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(1+\epsilon)^2-(1-2\epsilon)^2}\left(\lambda_{(1+\epsilon)}^{\delta}(p_\rho)\right.$$
$$\left. + i^t\lambda_{(1-2\epsilon)}^{\delta}(p_\rho)\right)^* n^*n\left(\lambda_{(1+\epsilon)}^{\delta}(p_\rho) + i^t\lambda_{(1-2\epsilon)}^{\delta}(p_\rho)\right)d(1+\epsilon)\, d(1$$
$$- 2\epsilon),$$

we conclude that

$$\eta(h_\rho n^*nh_\rho) = \delta_p(h_\rho n^*nh_\rho). \qquad (18)$$

Next let $n \in \vartheta_\delta$ be arbitrary. Using (18) and applying [6], Lemmme 7 (b) or [18], Proposition 1.1. we deduce for every ρ

$$\eta(h_\rho n^* n h_\rho) = \delta_p(h_\rho n^* n h_\rho) = \delta(\sqrt{p} h_\rho n^* n h_\rho \sqrt{p}) = \left\|\left(n h_\rho \sqrt{p}\right)_\delta\right\|^2 =$$
$$\left\|X_\delta \pi_\delta\left(p^{\delta}_{\frac{-i}{2}}(\sqrt{p}h_\rho)\right)X_\delta n_\delta\right\|^2 = \left\|X_\delta \pi_\delta(\sqrt{p})\pi_\delta\left(p^{\delta}_{\frac{-i}{2}}(h_\rho)\right)X_\delta n_\delta\right\|^2.$$

Since $h_\rho n^* n h_\rho \xrightarrow{\rho} n^* n$ and $p^{\delta}_{\frac{-i}{2}}(h_\rho) \xrightarrow{\rho} H_G$ in the T^*-topology, and η is lower semicontinuous in the T^*-topology, we get

$$\eta(n^* n) \le \lim_\rho \eta(h_\rho n^* n h_\rho)$$
$$= \lim_\rho \left\|X_\delta \pi_\delta(\sqrt{p})\pi_\delta\left(p^{\delta}_{\frac{-i}{2}}(h_\rho)\right)X_\delta n_\delta\right\|^2 = \left\|X_\delta \pi_\delta(\sqrt{p})X_\delta n_\delta\right\|^2.$$

Applying now [18], Corollary 1.2, we conclude:

$$\eta(n^* n) \le \left\|\left(n\sqrt{p}\right)_\delta\right\|^2 = \delta(\sqrt{p} n^* n \sqrt{p}) = \delta_p(n^* n). \quad (19)$$

To have (13) proved, we must show that (19) actually holds for every $n \in \vartheta_{\delta_p}$. This follows by the proof of [18], Lemma 2.1. We report it for sake of completeness.

For every $n \in \vartheta_\delta$, since $(1-2\epsilon)(p) \in G^\delta$ and $\vartheta_\delta G^\delta \subset \vartheta_\delta$, (19) yields

$$\eta\left(\left(H_G - (1-2\epsilon)(p)\right)n^* n\left(H_G - (1-2\epsilon)(p)\right)\right) \le \delta\left(\sqrt{p}\left(H_G - (1-2\epsilon)(p)\right)n^* n\left(H_G - (1-2\epsilon)(p)\right)\sqrt{p}\right) = 0.$$

ϑ_δ being ω-dense in G, it follows $\eta\left(H_G - (1-2\epsilon)(p)\right) = 0$, what means $(1-2\epsilon)(\eta) \le (1-2\epsilon)(p)$.

For $\epsilon \ge 0$ we consider the projection $e_{(1+2\epsilon)} = \nu_{\left[\frac{1}{(1+2\epsilon)}, +\infty\right)}(p) \in G^\delta$, where $\nu_{\left[\frac{1}{(1+2\epsilon)}, +\infty\right)}$ this depends on characteristic function of $\left[\frac{1}{(1+2\epsilon)}, +\infty\right)$. Then $e_{(1+2\epsilon)} \nearrow (1-2\epsilon)(p)$. We consider also the inverse $b_{(1+2\epsilon)}$ of $\sqrt{p}e_{(1+2\epsilon)}$ in the reduced algebra $e_{(1+2\epsilon)}G^\delta e_{(1+2\epsilon)}$: $q_{(1+2\epsilon)} = g_{(1+2\epsilon)}(p) \in G^\delta$ with

$g_{(1+2\epsilon)}(1+\epsilon) = \frac{1}{\sqrt{1+\epsilon}}\nu_{\left[\frac{1}{(1+2\epsilon)}, +\infty\right)}(1+\epsilon)$.

Now let $n \in \vartheta_{\delta_p}$ be arbitrary. Then $n\sqrt{p} \in \vartheta_\delta$, so

$$n e_{(1+2\epsilon)} = (n\sqrt{p})q_{(1+2\epsilon)} \in \vartheta_\delta G^\delta \subset \vartheta_\delta, \quad \epsilon \ge 0.$$

Applying (19) and [18], Corollary 1.2, we obtain for every $\epsilon \ge 0$

$$\eta\left(e_{(1+2\epsilon)} n^* n e_{(1+2\epsilon)}\right) \le \delta\left(\sqrt{p} e_{(1+2\epsilon)} n^* n e_{(1+2\epsilon)} \sqrt{p}\right) = \left\|\left(n e_{(1+2\epsilon)}\sqrt{p}\right)_\delta\right\|^2$$
$$= \left\|\left(n\sqrt{p}e_{(1+2\epsilon)}\right)_\delta\right\|^2 = \left\|X_\delta \pi_\delta\left(e_{(1+2\epsilon)}\right)X_\delta\left(n\sqrt{p}\right)_\delta\right\|^2.$$

Since $(1-2\epsilon)(\eta) \le (1-2\epsilon)(p), e_{(1+2\epsilon)} \nearrow (1-2\epsilon)(p)$ and η is lower semicontinuous in the T^*-topology, it follows

$$\begin{aligned}\eta(n^*n) &= \eta\big((1-2\epsilon)(p)n^*n(1-2\epsilon)(p)\big)\\ &\le \lim_{(1+2\epsilon)\to\infty} \eta\big(e_{(1+2\epsilon)}n^*ne_{(1+2\epsilon)}\big)\\ &\le \lim_{(1+2\epsilon)\to\infty} \left\|X_\delta \pi_\delta\big(e_{(1+2\epsilon)}\big)X_\delta\big(n\sqrt{p}\big)_\delta\right\|^2\\ &= \left\|X_\delta \pi_\delta\big((1-2\epsilon)(p)\big)X_\delta\big(n\sqrt{p}\big)_\delta\right\|^2.\end{aligned}$$

We apply [18], Corollary 1.2 again, we conclude:

$$\eta(n^*n) \le \left\|\Big(n\sqrt{p}(1-2\epsilon)(p)\Big)_\delta\right\|^2 = \left\|\big(n\sqrt{p}\big)_\delta\right\|^2 = \delta\big(\sqrt{p}n^*n\sqrt{p}\big) = \delta_p(n^*n).$$

Taking a λ^δ- invariant, hereditary*-subalgebra $\mathcal{H}_0$ of $\mathcal{H}_{\delta_p}$ so, the proof of the theorem will completed, then

$$\mathcal{H} \cap G^+ \subset \mathcal{H}_0 \cap G^+.$$
$$\eta(q) = \delta_p(q), \quad q \in \mathcal{H}_0 \cap G^+.$$

We notice that:

(i) $\left\{q \in \mathcal{H}_{\delta_p} \cap G^+; \eta(q) = \delta_p(q)\right\} \subset G$ is a face.

(ii) $\eta(q) = \delta_p(q)$ for all $q \in \mathcal{H} \cap G^+$.

Since $\left\{q \in \mathcal{H}_{\delta_p} \cap G^+; \eta(q) = \delta_p(q)\right\}$ is a convex cone, for (i) we have only to verify the implication

$$G^+ \ni q \le z \in \mathcal{H}_{\delta_p} \cap G^+; \eta(z) = \delta_p(z) \Longrightarrow \eta(q) = \delta_p(q). \qquad (20)$$

It follows surely by using

$$\eta(q) \le \delta_p(q), \qquad \eta(z-q) \le \delta_p(z-q),$$
$$\eta(q) + \eta(z-q) = \eta(z) = \delta_p(z) = \delta_p\,(q) + \delta_p(z-q) \le +\infty.$$

For (ii) let $q \in \mathcal{H} \cap G^+$ be arbitrary. Without loss of generality we can assume that $\|q\| \le 1$. Denoting $q_{(1+2\epsilon)} \coloneqq H_G - (H_G - q)^{(1+2\epsilon)} \in \mathcal{H} \cap {H_G}^+, \epsilon \ge 0$, we obtain an increasing sequence $\big(q_{(1+2\epsilon)}\big)_{\epsilon\ge 0}$ which is T^*-convergent to the support $(1-2\epsilon)(q)$ of q (see e.g. [15], Section 2.22). Since all $q_{(1+2\epsilon)}$ belong to the commutative C^*-subalgebra of G generated by q, the sequence $\big(q_{(1+2\epsilon)}qq_{(1+2\epsilon)}\big)_{\epsilon\ge 0}$ is still increasing and it is T^*-convergent to q. Therefore we deduce:

- $\eta\big(q_{(1+2\epsilon)}q_{(1+2\epsilon)}\big) = \delta_p\big(q_{(1+2\epsilon)}q_{(1+2\epsilon)}\big)$ for all $\epsilon \ge 0$ by the assumption on $\mathcal{H}$;
- $\eta\big(q_{(1+2\epsilon)}qq_{(1+2\epsilon)}\big) = \delta_p\big(q_{(1+2\epsilon)}qq_{(1+2\epsilon)}\big)$ for all $\epsilon \ge 0$ by applying (2.8) with $q = q_{(1+2\epsilon)}qq_{(1+2\epsilon)}$ and $z = q_{(1+2\epsilon)}q_{(1+2\epsilon)}$;

- $\eta(q) = \lim_{(1+2\epsilon)\to\infty} \eta\big(q_{(1+2\epsilon)} q q_{(1+2\epsilon)}\big) = \lim_{(1+2\epsilon)\to\infty} \delta_a\big(q_{(1+2\epsilon)} q q_{(1+2\epsilon)}\big) = \delta_p(q)$ by the normality of η and δ_p.

Now we set

$$\Omega_0 := \left\{q \in \mathcal{H}_{\delta_p}; 0 \le q \le z \text{ for some } z \in \mathcal{H} \cap G^+\right\},$$

$\vartheta_0 := \{m \in G;\ m^* m \in \Omega_0\}$,

$$\mathcal{H}_0 := \text{linear span of } \vartheta_0^* \vartheta_0.$$

Then Ω_0 is a face, $\mathcal{H}_0$ is p^*-subalgebra of , $\mathcal{H}_0 \cap G^+ = \Omega_0$, and $\mathcal{H}_0$ is the linear span of Ω_0 (see e.g. [15], Proposition 3.21).Thus $\mathcal{H}_0$ is a hereditary*-subalgebra of $\mathcal{H}_{\eta_p}$ and $\mathcal{H} \cap G^+ \subset \Omega_0 = \mathcal{H}_0 \cap G^+$. Since $\mathcal{H} \cap G^+$ is λ^δ- invariant, also Ω_0, and therefore $\mathcal{H}_0$ is λ^δ- invariant. Finally, the above (ii) and (i) imply that we have $\eta(q) = \delta_p(q)$ for all $q \in \Omega_0$.

Remark 7. If p is assumed only affiliated to G^δ and not necessarily bounded, the statement of Theorem 6 is not more true. Counterexamples can be obtained using [13], Proposition 7.8 or [6], Example 8.

Two faithful, semi-finite, normal weights η_0, η are constructed on $Y(\ell^2)$ such that $\eta_0 \le \eta$ and $\eta_0 \ne \eta$, but $\eta_0(m) = \eta(m)$ for $m \in \mathcal{H} \cap G^+$, where $\mathcal{H}$ is a $\omega -$dense*-subalgebra of $\mathcal{H}_\eta$ (in [6], Example 8, the construction delivers $\mathcal{H} = \mathcal{H}_\eta$).

Now let δ be a faithful, semi-finite, normal trace on $Y(\ell^2)$. By [13], Theorem 5.12 there exists a positive, self-adjoint operator P on ℓ^2, necessarily affiliated to $Y(\ell^2)^\delta = Y(\ell^2)$, such that $\eta_0 = \delta_P$. Then

- δ is a faithful, semi-finite, normal trace on $G = Y(\ell^2)$,
- P is a positive, self-adjoint operator to $G^\delta = Y(\ell^2)$,
- η is a λ^δ- invariant, faithful, semi-finite, normal weight on G,
- $\eta(m^* m) = \delta_P(m^* m)$ for $m \in \mathcal{H}$, where $\mathcal{H}$ is a $\omega -$dense*-subalgebra of $\mathcal{H}_\eta \subset \mathcal{H}_{\eta_0} = \mathcal{H}_{\delta_P}$,

but $\eta \nleq \delta_P$, because otherwise it would follow $\eta \le \delta_P = \eta_0$, hence $\eta = \eta_0$, in contradiction to $\eta \ne \eta_0$.

Remark 8. If in Theorem 6 we assume that $H_G - (1-2\epsilon)(\eta)$ belongs to the ω–closure of $\{n \in \mathcal{H}_\delta; n(1-2\epsilon)(\eta) = 0\}$ (that happens, for example, if $(1-2\epsilon)(\eta) \in G_\infty^\delta$, because $\mathcal{H}_\delta \mathcal{H}_\infty^\delta \subset \mathcal{H}_\delta$), then it follows also the equality $(1-2\epsilon)(\eta) = (1-2\epsilon)(p)$.

Since $(1-2\epsilon)(\eta) \le (1-2\epsilon)(p)$ trivially, we have to verify that for any $n \in \mathcal{H}_\delta$ with $n(1-2\epsilon)(\eta) = 0$, that is with $\eta(n^* n) = 0$, we have $n(1-2\epsilon)(p) = 0$.

By (16), by the lower semicontinuity of δ_p in the T^*-topology, and by (18), we obtain $\delta_p(n^* n) \le \lim_\rho \delta_p\big(h_\rho n^* n h_\rho\big) = \lim_\rho \eta\big(h_\rho n^* n h_\rho\big)$.

Using now the inequalities

$$\begin{aligned} h_\rho n^* n h_\rho &\le \left(2.H_G - h_\rho\right)n^* n\left(2.H_G - h_\rho\right) + h_\rho n^* n h_\rho \\ &= 2\left(\left(H_G - h_\rho\right)n^* n\left(H_G - h_\rho\right) + n^* n\right), \end{aligned}$$

and $\eta \le \delta_p$ as in [6], Lemme 7 (b) or [18], Proposition 1.1, we have

$$\begin{aligned} \delta_p(n^* n) &\le 2\lim_\rho \eta\left(\left(H_G - h_\rho\right)n^* n\left(H_G - h_\rho\right)\right) \le 2\lim_\rho \delta_p\left(\left(H_G - h_\rho\right)n^* n\left(H_G - h_\rho\right)\right) \\ &= 2\lim_\rho \left\|\left(n\left(H_G - h_\rho\right)\sqrt{p}\right)_\delta\right\|^2 \\ &= 2\lim_\rho \left\|X_\delta \pi_\delta\left(\lambda^\delta_{\frac{-i}{2}}\left(\sqrt{p}\left(H_G - h_\rho\right)\right)\right)X_\delta n_\delta\right\|^2 \\ &= 2\lim_\rho \left\|X_\delta \pi_\delta\left(\sqrt{p}\right)\pi_\delta\left(H_G - \lambda^\delta_{\frac{-i}{2}}\left(h_\rho\right)\right)X_\delta n_\delta\right\|^2. \end{aligned}$$

Since, by (16), $\lambda^\delta_{\frac{-i}{2}}\left(h_\rho\right) \xrightarrow{\rho} H_G$ in the T^*-topology, we conclude that $\delta_p(n^* n) = 0$, what is equivalent to $n\sqrt{p} = 0 \Leftrightarrow n(1-2\epsilon)(p) = 0$ [19].

The next theorem is a slight extension of [18], Theorem 2.3:

Theorem 9. Let G be a W^*-algebra, δ ,η a faithful, semi-finite, normal weight on $G, p \in \left(G^\delta\right)^+$, and η a λ^δ- invariant, normal weight on G. If there exists a ω-dense, λ^δ and λ^δ-invariant*- subalgebra $\mathcal{H}$ of $\mathcal{H}_{\delta_p}$ such that $\eta(m^* m) = \delta_p(m^* m)$, $m \in \mathcal{H}$,then $\eta = \delta_p$.

Proof. By Theorem 6 we have $\eta \le \delta_p$. In particular, η is semi-finite.

Addition to that, by [13],Theorem 5.12 there exists a positive, self-adjoint operator P, affiliated to G^δ, such that $\eta = \delta_P$. Since $\delta_P = \eta \le \delta_p$ [18], Lemma 2.2) yields $P \le p$. In particular, P is bounded.

Since $\mathcal{H}_{\delta_P}$ is the linear span of $\{q \in G^+ : \delta_P(q) < +\infty\}$, $\mathcal{H}_{\delta_p}$ is the linear span of $\left\{q \in G^+ : \delta_p(q) < +\infty\right\}$, and $\delta_P \le \delta_p$, we have $\mathcal{H} \subset \mathcal{H}_{\delta_p} \subset \mathcal{H}_{\eta_P}$. If we applying Theorem 6 again this leads us to deduce that $\delta_p \le \delta_P = \eta$. Theorem 10 is an equivalent and symmetric form of Theorem 9.

Theorem 10. Let G be a W^*-algebra, δ a faithful, semi-finite, normal weight on $G, p, q \in \left(G^\delta\right)^+$, and η a λ^δ- invariant, normal weight on G. If there exists a ω-dense, λ^δ and λ^δ-invariant*- subalgebra $\mathcal{H}$ of $\mathcal{H}_{\delta_p}$ then $\eta_q(m^* m) = \delta_p(m^* m)$, $m \in \mathcal{H}$,then $\eta_q = \delta_p$

Proof. Since η is λ^δ-invariant and $q \in \left(G^\delta\right)^+$, the normal weight η_q is still λ^δ-invariant : we have for every $(1+\epsilon) \in \mathbb{R}$ and $m \in G^+$

$$\begin{aligned} \eta_q\left(\lambda^\delta_{(1+\epsilon)}(m)\right) &= \eta\left(\sqrt{q}\lambda^\delta_{(1+\epsilon)}(m)\sqrt{q}\right) = \eta\left(\lambda^\delta_{(1+\epsilon)}\left(\sqrt{q}m\sqrt{q}\right)\right) = \eta\left(\sqrt{q}m\sqrt{q}\right) \\ &= \eta_q(m). \end{aligned}$$

Hence we applying Theorem 9 with η replaced by η_q. An immediate consequence of Theorem 2.4 an 2.5 is [13], Proposition 5.9 :

Corollary 11. Let G be a W^*-algebra, δ a faithful, semi-finite, normal weight on $G,$ and η a λ^δ-invariant, normal weight on G. If there exists a ω-dense, λ^δ, λ^δ-invariant*- subalgebra $\mathcal{H}$ of $\mathcal{H}_\delta$ such that $\eta(m^*m) = \delta(m^*m), \quad m \in \mathcal{H},$ then $\eta = \delta.$

Theorem 12. Let G be a W^*-algebra, δ, η a faithful, semi-finite, normal weights on $G, p, \in \left(G^\delta\right)^+, q \in \left(G^\delta\right)^+$. By assuming that there are a ω-dense, λ^δ-invariant*- subalgebra $\mathcal{H}_1$ of $\mathcal{H}_{\delta_p}$ and a ω-dense, λ^δ-invariant*- subalgebra $\mathcal{H}_2$ of $\mathcal{H}_{\eta_q}$ then $\eta_q(m^*m) = \delta_p(m^*m),\ m \in \mathcal{H}_1 \cup \mathcal{H}_2$. So, $\eta_q = \delta_p$.

Proof. We applying here twice Theorem 7. An immediate consequence of Theorem 12 are :

Theorem 13. Let G be a W^*-algebra, δ, η faithful, semi-finite, normal weights on $G,$ and $p \in \left(G^\delta\right)^+, q \in (G^{\,\eta})^+$. We assuming that there exists a ω-dense, both $\lambda^\delta -$ and λ^δ-invariant*- subalgebra $\mathcal{H}$ of $\mathcal{H}_{\delta_p} \cap \mathcal{H}_{\eta_q}$ then

$\eta_q(m^*m) = \delta_p(m^*m),\ m \in \mathcal{H}.$ Then $\eta_q = \delta_p$.

Corollary 14. Let G be a W^*-algebra, $\delta\,, \eta$ a faithful, semi-finite, normal weight on G. If there exists a ω-dense, both λ^δ and λ^δ-invariant*- subalgebra $\mathcal{H}$ of $\mathcal{H}_{\delta_p} \cap \mathcal{H}_{\eta_p}$ such that $\eta(m^*m) = \delta(m^*m), \quad m \in \mathcal{H},$ then $\eta = \delta$.There exist also criteria of different kind for equality and inequalities between faithful, semi-infinite, normal weights, due to [5]. They are in trems of the Connes cocycle (see [5], Section 1.2 or [15], Theorem 10.28 and C.10.4): if δ and η are faithful, semi-finite, normal weights a W^*-algebra, the Connes cocycle of η with respect to δ will be denoted by $(U\eta\colon U\delta)_{(1+\epsilon)}\Gamma(U\eta\colon U\delta)_\Gamma \in G,$ it is analytic in the interior and satisfies $\left\|(U\eta\colon U\delta)_{-\frac{1}{2}}\right\| \leq 1.$

(i) $\eta(p) = \delta(p)$ for all $p \in \Omega_\delta$ if and only if $\mathbb{R} \ni (1+\epsilon) \longmapsto (U\eta\colon U\delta)_{(1+\epsilon)} \in G$ has a $\omega -$continuous extension $\left\{\Gamma \in \mathbb{C}; -\frac{1}{2} \leq \mathrm{lm}\,\Gamma \leq 1\right\} \ni \Gamma \longmapsto (U\eta\colon U\delta)_\Gamma \in G,$ which is analytic in the interior and such that $(U\eta\colon U\delta)_{-\frac{1}{2}}$ is isometric.

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