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RESEARCH ARTICLE

NEW SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS ASSOCIATED WITH A SYMMETRIC DIFFERENTIAL OPERATOR

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Abstract

In this paper, we introduce a new subclass of harmonic univalent functions in the open unit disk U by using a symmetric differential operator. Properties for the class $Q_H(m,\lambda_1,\lambda_2,\beta,\delta)$ and $Q_{\hat{H}}(m,\lambda_1,\lambda_2,\beta,\delta)$ are established.

1. Introduction:

Let u,v be real harmonic functions in a simply connected domain E, then the continuous complex-valued function $f=u+iv$ is said to be harmonic in E. In any simply connected domain $E\subset\mathbb{C}$, we can write

$$f(z)=h(z)+\overline{g(z)}, \tag{1.1}$$

where f and g are analytic in E. We call h the analytic part and g co-analytic part of f. The function is sense preserving and univalent in E, if the Jacobian of f, $J_{f(z)}=|h'(z)|-|g'(z)|>0$ see [7]. Let H denote the class of functions of the form (1.1), which are harmonic, univalent and sense-preserving in the open unit disk $U=\{z:|z|<1\}$ with $f(0)=h(0)=0$ and $f_z(0)=1$. We define

$$h(z)=z+\sum_{n=2}^{\infty}a_nz^n,\quad g(z)=\sum_{n=1}^{\infty}b_nz^n,\quad |b_1|<1. \tag{1.2}$$

Note that, if the co-analytic of f is zero, then the class H reduces to the class of normalized analytic functions.

Also let $\hat{H}$ denote the subclass of H consisting of functions $f=h+\overline{g}$ in the form

$$h(z)=z-\sum_{n=2}^{\infty}|a_n|z^n,\quad g(z)=\sum_{n=1}^{\infty}|b_n|z^n,\quad |b_1|<1. \tag{1.3}$$

Definition 1: Let $f\in A$, which is analytic and univalent in U. For a function $f(z)$, we formula the symmetric differential operator as follows:

$$S^0_{\lambda_1,\lambda_2}f(z)-f(z)=0$$

$$S^1_{\lambda_1,\lambda_2}f(z)-(1-\lambda_1)f(z)=(\lambda_1-\lambda_2)zf'(z)-\lambda_2zf'(-z)$$

$$=(1-\lambda_1)\left(z+\sum_{n=2}^{\infty}a_nz^n\right)+(\lambda_1-\lambda_2)\left(z+\sum_{n=2}^{\infty}n\,a_nz^n\right)$$

$$-\lambda_2\left(-z+\sum_{n=2}^{\infty}n\,a_n(-1)^nz^n\right)$$

$$=z+\sum_{n=2}^{\infty}\left[n\left(\lambda_1-\lambda_2\left(1+(-1)^n\right)\right)+1-\lambda_1\right]a_nz^n$$

$$S^2_{\lambda_1,\lambda_2}f(z)=S^1_{\lambda_1,\lambda_2}\left(S^1_{\lambda_1,\lambda_2}f(z)\right)=z+\sum_{n=2}^{\infty}\left[n\left(\lambda_1-\lambda_2\left(1+(-1)^n\right)\right)+1-\lambda_1\right]^2a_nz^n$$

$$\vdots$$

$$S^m_{\lambda_1,\lambda_2}f(z)=z+\sum_{n=2}^{\infty}\left[n\left(\lambda_1-\lambda_2\left(1+(-1)^n\right)\right)+1-\lambda_1\right]^m a_nz^n. \tag{1.4}$$

For $\lambda_1\geq 0,\ \lambda_2\leq\lambda_1$. We not that when $\lambda_2=0$, we have Al-Oboudi differential operator [1], we may say that (1.2) is the symmetric Al-Oboudi differential operator, and the symmetric Al-Oboudi integral operator $\Im^m_{\lambda_1,\lambda_2}$ will be as:

$$\Im^m_{\lambda_1,\lambda_2}f(z)=z+\sum_{n=2}^{\infty}\frac{1}{\left[n\left(\lambda_1-\lambda_2\left(1+(-1)^n\right)\right)+1-\lambda_1\right]^m}a_nz^n. \tag{1.5}$$

We also not that when $\lambda_1=1$ in (1,4) and (1.5), we have the symmetric Salagean differential and integral operator respectively, studied by W. Ibrahim and M. Darus [3].

For $f=h+\overline{g}$ given by (1.2), we define the operator $S^m_{\lambda_1,\lambda_2}$ as:

$$S^m_{\lambda_1,\lambda_2}f(z)=S^m_{\lambda_1,\lambda_2}h(z)+(-1)^m\overline{S^m_{\lambda_1,\lambda_2}g(z)},\ \lambda_1\geq 0,\ \lambda_1\geq\lambda_2,\ m\in N_0, \tag{1.6}$$

such that $S^m_{\lambda_1,\lambda_2}h(z)=z+\sum_{n=2}^{\infty}\left[n\left(\lambda_1-\lambda_2\left(1+(-1)^n\right)\right)+1-\lambda_1\right]^m a_nz^n$,

$$S^m_{\lambda_1,\lambda_2} g(z) = \sum_{n=1}^{\infty} \left[n\left(\lambda_1 - \lambda_2\left(1+(-1)^n\right)\right) + 1 - \lambda_1 \right]^m b_n z^n.$$

Definition 2: A function $f \in H$ is said to be in the class $Q_H(m,\lambda_1,\lambda_2,\beta,\delta)$ if it satisfies the following condition

$$\operatorname{Re}\left\{ \left(1+\left(e^{i\theta}+\beta\right)\right) \frac{S^{m+1}_{\lambda_1,\lambda_2} f(z)}{S^{m}_{\lambda_1,\lambda_2} f(z)} - \left(e^{i\theta}+\beta\right) \right\} > \delta, \quad 0 \le \delta < 1, \quad \beta \ge 0, \ \theta \in \mathbb{R}, \tag{1.7}$$

where $S^m_{\lambda_1,\lambda_2} f(z)$ is defined by (1,6).

Also, we let the subclass $Q_{\hat{H}}(m,\lambda_1,\lambda_2,\beta,\delta)$ consists of harmonic functions $f_m = h + \overline{g_m}$ such that h and g_m are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \qquad g_m(z) = (-1)^m \sum_{n=1}^{\infty} |b_n| z^n. \tag{1.8}$$

We note that the class $Q_{\hat{H}}(m,\lambda_1,0,0,\delta) \equiv M\hat{H}(n,\lambda,\gamma)$ introduced and studied by Yalcin S. *et al*. [8]. when $Q_{\hat{H}}(0,1,0,0,\delta) \equiv G_{\hat{H}}(\gamma)$ which is defined and studied by Rosy *et al*. [4].

In this paper, we will give sufficient condition for functions f given by (1.1) to be in the class $Q_H(m,\lambda_1,\lambda_2,\beta,\delta)$ and it is shown that this coefficient condition is also necessary for functions in the class $Q_{\hat{H}}(m,\lambda_1,\lambda_2,\beta,\delta)$. Also we obtain distortion theorem and the extreme points for functions in the class $Q_{\hat{H}}(m,\lambda_1,\lambda_2,\beta,\delta)$.

2. Coefficient bound

We first begin with a sufficient condition for function $f(z)$ of the form (1.1), and for function $f_m(z)$ of the form (1.8), to be in the classes $Q_H(m,\lambda_1,\lambda_2,\beta,\delta)$ and $Q_{\hat{H}}(m,\lambda_1,\lambda_2,\beta,\delta)$ respectively.

Theorem 2.1 Let $f = h + \overline{g}$ be given by (1.1). If

$$\sum_{n=1}^{\infty} \Phi^m \left\{ (2\Phi-\delta-1)+(\Phi-1)\beta |a_n| + (2\Phi+\delta+1)+(\Phi+1)\beta |b_n| \right\} \le 2(1-\delta), \tag{2.1}$$

where $\Phi^m = \left[n\left(\lambda_1 - \lambda_2\left(1+(-1)^n\right)\right) + 1 - \lambda_1 \right]^m$, $\beta \ge 0$, and $0 \le \delta < 1$, then f is sense preserving, harmonic univalent in U and $f \in Q_H(m,\lambda_1,\lambda_2,\beta,\delta)$.

Proof: first, we prove that f is sense preserving and univalent in U.

Since $n \le \Phi^m \left\{ (2\Phi-\delta-1)+(\Phi-1)\beta \right\} / 2(1-\delta)$ and $n \le \Phi^m \left\{ (2\Phi+\delta+1)+(\Phi+1)\beta \right\} / 2(1-\delta)$,

$$\begin{aligned} |h'(z)| &\ge 1 - \sum_{n=2}^{\infty} n |a_n| |z|^{n-1} \\ &\ge 1 - \sum_{n=2}^{\infty} \frac{\Phi^m \left\{ (2\Phi-\delta-1)+(\Phi-1)\beta \right\}}{1-\delta} |a_n| \\ &\ge \sum_{n=1}^{\infty} \frac{\Phi^m \left\{ (2\Phi+\delta+1)+(\Phi+1)\beta \right\}}{1-\delta} |b_n| \\ &\ge \sum_{n=1}^{\infty} n |b_n| > \sum_{n=1}^{\infty} n |b_n| |z|^{n-1} \ge |g'(z)| \end{aligned}$$

which shows f is sense preserving.

Next, If $z_1 \ne z_2$, then

$$\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| \geq 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right|$$

$$=1-\left|\frac{\sum_{n=1}^{\infty} b_n\left(z_1^n-z_2^n\right)}{(z_1-z_2)+\sum_{n=2}^{\infty} a_n\left(z_1^n-z_2^n\right)}\right|$$

$$>1-\frac{\sum_{n=1}^{\infty} n|b_n|}{1-\sum_{n=2}^{\infty} n|a_n|}$$

$$\geq 1-\frac{\sum_{n=1}^{\infty}\frac{\Phi^m\{(2\Phi+\delta+1)+(\Phi+1)\beta\}}{1-\delta}|b_n|}{1-\sum_{n=2}^{\infty}\frac{\Phi^m\{(2\Phi-\delta-1)+(\Phi-1)\beta\}}{1-\delta}|a_n|}$$

$$\geq 0$$

which proves univalence.

Finally, we show that $f \in Q_H(m,\lambda_1,\lambda_2,\beta,\delta)$. By using the fact that $\operatorname{Re} w \geq \delta$ if and only if, it suffices to show that

$$\left|(1-\delta)S^m_{\lambda_1,\lambda_2}f(z)+\left(1+\left(e^{i\theta}+\beta\right)\right)S^{m+1}_{\lambda_1,\lambda_2}f(z)-\left(e^{i\theta}+\beta\right)S^m_{\lambda_1,\lambda_2}f(z)\right| \\ -\left|(1+\delta)S^m_{\lambda_1,\lambda_2}f(z)-\left(1+\left(e^{i\theta}+\beta\right)\right)S^{m+1}_{\lambda_1,\lambda_2}f(z)+\left(e^{i\theta}+\beta\right)S^m_{\lambda_1,\lambda_2}f(z)\right| \tag{2.2}$$

Substituting the value of $S^m_{\lambda_1,\lambda_2}f(z)$ and $S^{m+1}_{\lambda_1,\lambda_2}f(z)$ in (1,5) yields, by

$$\left|\left(1-\delta-e^{i\theta}-\beta\right)S^m_{\lambda_1,\lambda_2}f(z)+\left(1+\left(e^{i\theta}+\beta\right)\right)S^{m+1}_{\lambda_1,\lambda_2}f(z)\right| \\ -\left|-\left(1+\delta+e^{i\theta}+\beta\right)S^m_{\lambda_1,\lambda_2}f(z)+\left(1+\left(e^{i\theta}+\beta\right)\right)S^{m+1}_{\lambda_1,\lambda_2}f(z)\right|$$

$$=\left|(2-\delta)z+\sum_{n=2}^{\infty}\left(1-\delta+\Phi+\Phi e^{i\theta}+\Phi\beta-e^{i\theta}-\beta\right)\Phi^m a_n z^n \right. \\ \left. -(-1)^m\sum_{n=2}^{\infty}\left(\delta-1+\Phi+\Phi e^{i\theta}+\Phi\beta+e^{i\theta}+\beta\right)\Phi^m\overline{b_n z^n}\right| \\ -\left|\delta z-\sum_{n=2}^{\infty}\left(\Phi+\Phi e^{i\theta}+\Phi\beta-1-\delta-e^{i\theta}-\beta\right)\Phi^m a_n z^n \right. \\ \left. +(-1)^m\sum_{n=2}^{\infty}\left(\Phi+\Phi e^{i\theta}+\Phi\beta+\delta+1+e^{i\theta}+\beta\right)\Phi^m\overline{b_n z^n}\right|$$

$$=\left|-\delta z-\sum_{n=2}^{\infty}\left(\Phi+\Phi e^{i\theta}+\Phi\beta-1-\delta-e^{i\theta}-\beta\right)\Phi^m a_n z^n \right. \\ \left. +(-1)^m\sum_{n=2}^{\infty}\left(\Phi+\Phi e^{i\theta}+\Phi\beta+\delta+1+e^{i\theta}+\beta\right)\Phi^m\overline{b_n z^n}\right|$$

$$\ge (2-\delta)|z| \sum_{n=2}^{\infty} \{(2\Phi-\delta)+(\Phi-1)\beta\} \Phi^m |a_n||z|^n$$
$$-\sum_{n=2}^{\infty} \{(2\Phi+\delta)+(\Phi+1)\beta\} \Phi^m |b_n||z|^n$$
$$-\delta|z| - \sum_{n=2}^{\infty} \{(2\Phi-\delta-2)+(\Phi-1)\beta\} \Phi^m |a_n||z|^n$$
$$-\sum_{n=2}^{\infty} \{(2\Phi+\delta+2)+(\Phi+1)\beta\} \Phi^m |b_n||z|^n$$
$$= 2(1-\delta)|z| - 2\sum_{n=2}^{\infty} \{(2\Phi-\delta-1)+(\Phi-1)\beta\} \Phi^m |a_n||z|^n$$
$$-2\sum_{n=2}^{\infty} \{(2\Phi+\delta+1)+(\Phi+1)\beta\} \Phi^m |b_n||z|^n$$
$$= 2(1-\delta)|z| \left\{1 - \sum_{n=2}^{\infty} \frac{\Phi^m \{(2\Phi-\delta-1)+(\Phi-1)\beta\}}{1-\delta} |a_n||z|^{n-1}\right.$$
$$\left. - \sum_{n=2}^{\infty} \frac{\Phi^m \{(2\Phi+\delta+1)+(\Phi+1)\beta\}}{1-\delta} |b_n||z|^{n-1}\right\}$$
$$> 2(1-\delta)|z| \left\{1 - \sum_{n=2}^{\infty} \frac{\Phi^m \{(2\Phi-\delta-1)+(\Phi-1)\beta\}}{1-\delta} |a_n|\right.$$
$$\left. - \sum_{n=2}^{\infty} \frac{\Phi^m \{(2\Phi+\delta+1)+(\Phi+1)\beta\}}{1-\delta} |b_n|\right\}$$

≥ 0, by (1.4).

The harmonic function

$$f(z) = z + \sum_{n=2}^{\infty} \frac{1-\delta}{\Phi^m \{(2\Phi-\delta-1)+(\Phi-1)\beta\}} x_n z^n + \sum_{n=1}^{\infty} \frac{1-\delta}{\Phi^m \{(2\Phi-\delta-1)+(\Phi-1)\beta\}} \overline{y}_n \overline{z}^n, \qquad (2.3)$$

where $\Phi^m = \left[n\left(\lambda_1 - \lambda_2\left(1+(-1)^n\right)\right)+1-\lambda_1\right]^m$, $\lambda_1 \ge 0$, $\lambda_1 \ge \lambda_2$, $m \in N_0$ and $\sum_{n=1}^{\infty}|x_n| + \sum_{n=2}^{\infty}|y_n| = 1$,

shows that the coefficient bound given by (2.1) is sharp.

The functions of the form (2.3) are in $Q_H(m,\lambda_1,\lambda_2,\beta,\delta)$ because

$$\sum_{n=1}^{\infty}\left(\frac{\Phi^m \{(2\Phi-\delta-1)+(\Phi-1)\beta\}}{1-\delta}|a_n| + \frac{\Phi^m \{(2\Phi+\delta+1)+(\Phi+1)\beta\}}{1-\delta}|b_n|\right) = 1 + \sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n| = 2.$$

In the following theorem, it is shown that the condition (2.1) is also necessary for functions $f_m = h + \overline{g}_m$, where h and g_m are of the form (1.8).

Theorem 2.2: Let $f_m = h + \overline{g}_m$ be given by (1.8). Then $f_m \in Q_{\hat{H}}(m,\lambda_1,\lambda_2,\beta,\delta)$ if and only if

$$\sum_{n=1}^{\infty} \Phi^m \{(2\Phi-\delta-1)+(\Phi-1)\beta|a_n| + (2\Phi+\delta+1)+(\Phi+1)\beta|b_n|\} \le 2(1-\delta) \qquad (2.4)$$

where $\Phi^m = \left[n\left(\lambda_1 - \lambda_2\left(1+(-1)^n\right)\right)+1-\lambda_1\right]^m$, $\beta \ge 0$, and $0 \le \delta < 1$,

Proof: Since $Q_{\hat{H}}(m,\lambda_1,\lambda_2,\beta,\delta) \subset Q_H(m,\lambda_1,\lambda_2,\beta,\delta)$ we only need to prove "only if " part of Theorem 2.2. To this end, for functions f_m of the form (1.8), we notice that the condition (1.7) is equivalent to

$$\operatorname{Re}\left\{\left(1+\left(e^{i\theta}+\beta\right)\right)\frac{S^{m+1}_{\lambda_1,\lambda_2}f(z)}{S^{m}_{\lambda_1,\lambda_2}f(z)}-\left(e^{i\theta}+\beta+\delta\right)\right\}$$

$$=\operatorname{Re}\left\{\frac{\left(1+\left(e^{i\theta}+\beta\right)\right)S^{m+1}_{\lambda_1,\lambda_2}f(z)-\left(e^{i\theta}+\beta+\delta\right)S^{m}_{\lambda_1,\lambda_2}f(z)}{S^{m}_{\lambda_1,\lambda_2}f(z)}\right\}$$

$$=\operatorname{Re}\left\{\frac{\left(1+\left(e^{i\theta}+\beta\right)\right)\left(z-\sum_{n=2}^{\infty}\Phi^{m+1}|a_n|z^n+(-1)^{2m}\sum_{n=1}^{\infty}\Phi^{m+1}|b_n|\overline{z}^n\right.}{z-\sum_{n=2}^{\infty}\Phi^{m}|a_n|z^n+(-1)^{2m}\sum_{n=1}^{\infty}\Phi^{m}|b_n|\overline{z}^n}\right.$$

$$\left.-\frac{\left(e^{i\theta}+\beta+\delta\right)\left(z-\sum_{n=2}^{\infty}\Phi^{m}|a_n|z^n+(-1)^{2m}\sum_{n=1}^{\infty}\Phi^{m}|b_n|\overline{z}^n\right.}{z-\sum_{n=2}^{\infty}\Phi^{m}|a_n|z^n+(-1)^{2m}\sum_{n=1}^{\infty}\Phi^{m}|b_n|\overline{z}^n}\right\}$$

$$=\operatorname{Re}\left\{\frac{(1-\delta)z-\sum_{n=2}^{\infty}\Phi^{m}\left\{\Phi\left(1+e^{i\theta}+\beta\right)-\left(e^{i\theta}+\beta+\delta\right)\right\}|a_n|z^n}{z-\sum_{n=2}^{\infty}\Phi^{m}|a_n|z^n+(-1)^{2m}\sum_{n=1}^{\infty}\Phi^{m}|b_n|\overline{z}^n}\right.$$

$$\left.-\frac{(-1)^{2m}\sum_{n=2}^{\infty}\Phi^{m}\left\{\Phi\left(1+e^{i\theta}+\beta\right)+\left(e^{i\theta}+\beta+\delta\right)\right\}|b_n|\overline{z}^n}{z-\sum_{n=2}^{\infty}\Phi^{m}|a_n|z^n+(-1)^{2m}\sum_{n=1}^{\infty}\Phi^{m}|b_n|\overline{z}^n}\right\}$$

$$=\operatorname{Re}\left\{\frac{1-\delta-\sum_{n=2}^{\infty}\Phi^{m}\left\{\Phi-\delta+(\Phi-1)e^{i\theta}+(\Phi-1)\beta\right\}|a_n|z^{n-1}}{1-\sum_{n=2}^{\infty}\Phi^{m}|a_n|z^n+\frac{\overline{z}}{z}(-1)^{2m}\sum_{n=1}^{\infty}\Phi^{m}|b_n|\overline{z}^{n-1}}\right.$$

$$\left.-\frac{(-1)^{2m}\sum_{n=2}^{\infty}\Phi^{m}\left\{\Phi+\delta+(\Phi+1)e^{i\theta}+(\Phi+1)\beta\right\}|b_n|\overline{z}^{n-1}}{1-\sum_{n=2}^{\infty}\Phi^{m}|a_n|z^n+\frac{\overline{z}}{z}(-1)^{2m}\sum_{n=1}^{\infty}\Phi^{m}|b_n|\overline{z}^{n-1}}\right\}$$

$$\geq 0.$$

The above condition must hold for all values of z, $|z|=r<1$, we must have

$$=\operatorname{Re}\left\{\frac{1-\delta-\sum_{n=2}^{\infty}\Phi^{m}\left\{\Phi-\delta+(\Phi-1)\beta\right\}|a_n|r^{n-1}-\sum_{n=1}^{\infty}\Phi^{m}\left\{\Phi+\delta+(\Phi+1)\beta\right\}|b_n|r^{n-1}}{1-\sum_{n=2}^{\infty}\Phi^{m}|a_n|z^n+\sum_{n=1}^{\infty}\Phi^{m}|b_n|r^{n-1}}\right.$$

$$\left.-e^{i\theta}\frac{\sum_{n=2}^{\infty}\Phi^{m}(\Phi-1)|a_n|r^{n-1}+\sum_{n=1}^{\infty}\Phi^{m}(\Phi+1)|b_n|r^{n-1}}{1-\sum_{n=2}^{\infty}\Phi^{m}|a_n|r^n+\sum_{n=1}^{\infty}\Phi^{m}|b_n|r^{n-1}}\right\}\geq 0$$

Since $\operatorname{Re}\left(-e^{i\theta}\right)\geq-\left|e^{i\theta}\right|=-1$, the above inequality reduce to

$$\frac{1-\delta-\sum_{n=2}^{\infty}\Phi^{m}\left\{(2\Phi-\delta-1)+(\Phi-1)\beta\right\}|a_n|r^{n-1}-\sum_{n=1}^{\infty}\Phi^{m}\left\{(2\Phi+\delta+1)+(\Phi+1)\beta\right\}|b_n|r^{n-1}}{1-\sum_{n=2}^{\infty}\Phi^{m}|a_n|z^{n}+\sum_{n=1}^{\infty}\Phi^{m}|b_n|r^{n-1}}\geq 0 \qquad (2.5)$$

In the condition (2.4) does not hold then the number in (2.5) is negative for r sufficiently close to 1. Thus there exists a $z_0=r_0$ in $(0,1)$ for which the quotient in (2.5) is negative. This contradicts the condition for $f_m\in Q_{\hat{H}}(m,\lambda_1,\lambda_2,\beta,\delta)$ and hence the result.

3. Distortion bounds

In the following theorem we will give the distortion bound for functions in $Q_{\hat{H}}(m,\lambda_1,\lambda_2,\beta,\delta)$.

Theorem 3.1: Let $f_m=h+\overline{g}_m$ be given by (1.8). Then for $|z|=r<1$ we have

$$|f_m|\leq(1+|b_1|)r+\frac{1-\delta}{(\lambda_1-4\lambda_2+1)^{m}\left\{(2\lambda_1-8\lambda_2-\delta+1)+(\lambda_1-4\lambda_2+1)\beta\right\}}\times\left(1-\frac{3+\delta+2\beta}{1-\delta}|b_1|\right)r^2,$$

$$|f_m|\geq(1+|b_1|)r-\frac{1-\delta}{(\lambda_1-4\lambda_2+1)^{m}\left\{(2\lambda_1-8\lambda_2-\delta+1)+(\lambda_1-4\lambda_2+1)\beta\right\}}\times\left(1-\frac{3+\delta+2\beta}{1-\delta}|b_1|\right)r^2.$$

Proof: we only prove the right hand inequality. The proof for the lift hand inequality is similar and will be omitted. Let $f_m\in Q_{\hat{H}}(m,\lambda_1,\lambda_2,\beta,\delta)$. Taking the absolute value of f_m we obtain

$$|f_m|=\left|z-\sum_{n=2}^{\infty}a_n z^{n}+(-1)^{m}\sum_{n=1}^{\infty}b_n z^{n}\right|$$

$$\leq(1+|b_1|)r+\sum_{n=2}^{\infty}(|a_n|+|b_n|)r^{n}$$

$$\leq(1+|b_1|)r+\frac{1-\delta}{(\lambda_1-4\lambda_2+1)^{m}\left\{(2\lambda_1-8\lambda_2-\delta+1)+(\lambda_1-4\lambda_2+1)\beta\right\}}$$
$$\times\sum_{n=1}^{\infty}\left(\frac{\Phi^{m}\left\{(2\Phi-\delta-1)+(\Phi-1)\beta\right\}}{1-\delta}|a_n|+\frac{\Phi^{m}\left\{(2\Phi+\delta+1)+(\Phi+1)\beta\right\}}{1-\delta}|b_n|\right)r^2$$

$$\leq(1+|b_1|)r+\frac{1-\delta}{(\lambda_1-4\lambda_2+1)^{m}\left\{(2\lambda_1-8\lambda_2-\delta+1)+(\lambda_1-4\lambda_2+1)\beta\right\}}\times\left(1-\frac{3+\delta+2\beta}{1-\delta}|b_1|\right)r^2$$

The functions

$$f(z)=z+|b_1|\overline{z}+\frac{1}{(\lambda_1-4\lambda_2+1)^m}\left[\frac{1-\delta}{\{(2\lambda_1-8\lambda_2-\delta+1)+(\lambda_1-4\lambda_2+1)\beta\}}-\frac{3+\delta+2\beta}{\{(2\lambda_1-8\lambda_2-\delta+1)+(\lambda_1-4\lambda_2+1)\beta\}}|b_1|\right]\overline{z}^2$$

$$f(z)=(1-|b_1|)z-\frac{1}{(\lambda_1-4\lambda_2+1)^m}\left[\frac{1-\delta}{\{(2\lambda_1-8\lambda_2-\delta+1)+(\lambda_1-4\lambda_2+1)\beta\}}-\frac{3+\delta+2\beta}{\{(2\lambda_1-8\lambda_2-\delta+1)+(\lambda_1-4\lambda_2+1)\beta\}}|b_1|\right]z^2$$

for $|b_1|\le\frac{1-\delta}{3+\delta+2\beta}$ shows that the bounds given in Theorem 3.1 are sharp.

The following covering result follows from the left hand inequality in Theorem 3.1

Corollary 3.2: Let $f_m=h+\overline{g}_m$ be given by (1.8). Then $f_m\in Q_{\hat{H}}(m,\lambda_1,\lambda_2,\beta,\delta)$. Then

$$\left\{w:|w|<\frac{(\lambda_1-4\lambda_2+1)^m\{(2\lambda_1-8\lambda_2-\delta+1)+(\lambda_1-4\lambda_2+1)\beta\}-(1-\delta)}{(\lambda_1-4\lambda_2+1)^m\{(2\lambda_1-8\lambda_2-\delta+1)+(\lambda_1-4\lambda_2+1)\beta\}}-\frac{(3+\delta+\beta)-(\lambda_1-4\lambda_2+1)^m\{(2\lambda_1-8\lambda_2-\delta+1)+(\lambda_1-4\lambda_2+1)\beta\}}{(\lambda_1-4\lambda_2+1)^m\{(2\lambda_1-8\lambda_2-\delta+1)+(\lambda_1-4\lambda_2+1)\beta\}}|b_1|\right\}\subset f_m(U).$$

4. Extreme points

In the following Theorem we determine the extreme points of $Q_{\hat{H}}(m,\lambda_1,\lambda_2,\beta,\delta)$.

Theorem 4.1: Let $f_m=h+\overline{g}_m$ be given by (1.8). Then $f_m\in Q_{\hat{H}}(m,\lambda_1,\lambda_2,\beta,\delta)$, if and only if

$$f_m(z)=\sum_{n=1}^{\infty}\left(X_n h_n(z)+y_n g_{k_n}(z)\right) \tag{4.1}$$

where $h_1(z)=z$,

$$h_n(z)=z-\frac{1-\delta}{\Phi^m\{(2\Phi-\delta-1)+(\Phi-1)\beta\}}z^n \qquad (n=2,3,\ldots),$$

$$g_{m_n}=z+(-1)^m\frac{1-\delta}{\Phi^m\{(2\Phi+\delta+1)+(\Phi+1)\beta\}}\overline{z}^n \qquad (n=1,2,\ldots),$$

$$\sum_{n=1}^{\infty}(X_n+Y_n)=1,\quad X_n\ge0, Y_n\ge0.$$

In particular, the extreme points of $Q_{\hat{H}}(m,\lambda_1,\lambda_2,\beta,\delta)$ are $\{h_n\}$ and $\{g_{m_n}\}$

Proof: For functions f_m of the form (4.1) we have

$$f_m(z)=\sum_{n=1}^{\infty}\left(X_n h_n(z)+Y_n g_{m_n}(z)\right)$$

$$=\sum_{n=1}^{\infty}\left(X_n+Y_n\right)z-\sum_{n=2}^{\infty}\frac{1-\delta}{\Phi^m\left\{\left(2\Phi-\delta-1\right)+\left(\Phi-1\right)\beta\right\}}X_n z^n$$

$$+\left(-1\right)^m\sum_{n=1}^{\infty}\frac{1-\delta}{\Phi^m\left\{\left(2\Phi+\delta+1\right)+\left(\Phi+1\right)\beta\right\}}Y_n\overline{z}^n$$

Then

$$\sum_{n=2}^{\infty}\frac{\Phi^m\left\{\left(2\Phi-\delta-1\right)+\left(\Phi-1\right)\beta\right\}}{1-\delta}\left|a_n\right|+\sum_{n=1}^{\infty}\frac{\Phi^m\left\{\left(2\Phi+\delta+1\right)+\left(\Phi+1\right)\beta\right\}}{1-\delta}\left|b_n\right|$$

$$=\sum_{n=2}^{\infty}X_n+\sum_{n=1}^{\infty}Y_n=1-X_1\le 1$$

and so $f_m\in Q_{\hat{H}}\left(m,\lambda_1,\lambda_2,\beta,\delta\right)$,

Conversely, suppose that $f_m\in Q_{\hat{H}}\left(m,\lambda_1,\lambda_2,\beta,\delta\right)$, Setting

$$X_n=\frac{\Phi^m\left\{\left(2\Phi-\delta-1\right)+\left(\Phi-1\right)\beta\right\}}{1-\delta}a_n,\qquad\left(n=2,3,\ldots\right),$$

$$Y_n=\frac{\Phi^m\left\{\left(2\Phi+\delta+1\right)+\left(\Phi+1\right)\beta\right\}}{1-\delta}b_n,\qquad\left(n=1,2,\ldots\right),$$

and $X_1=1-\sum_{n=2}^{\infty}X_n+\sum_{n=1}^{\infty}Y_n$ then f_m can be written as

$$f_m\left(z\right)=z-\sum_{n=2}^{\infty}\left|a_n\right|z^n+\left(-1\right)^n\sum_{n=1}^{\infty}\left|b_n\right|\overline{z}^n$$

$$=z-\frac{\left(1-\delta\right)X_n}{\Phi^m\left\{\left(2\Phi-\delta-1\right)+\left(\Phi-1\right)\beta\right\}}z^n$$

$$+\left(-1\right)^m\frac{\left(1-\delta\right)Y_n}{\Phi^m\left\{\left(2\Phi+\delta+1\right)+\left(\Phi+1\right)\beta\right\}}\overline{z}^n$$

$$=z+\sum_{n=2}^{\infty}\left(h_n\left(z\right)-z\right)X_n+\sum_{n=1}^{\infty}\left(g_{n_m}\left(z\right)-z\right)Y_n$$

$$=\sum_{n=2}^{\infty}h_n\left(z\right)X_n+\sum_{n=1}^{\infty}g_{n_m}\left(z\right)Y_n+z\left(1-\sum_{n=2}^{\infty}X_n+\sum_{n=1}^{\infty}Y_n\right)$$

$$=\sum_{n=1}^{\infty}\left(h_n\left(z\right)X_n+g_{n_m}\left(z\right)Y_n\right).$$

As required.

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