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RESEARCH ARTICLE

SPHERICAL AND ENDPOINT BOUNDS FOR DERIVATIVES OF FRACTIONAL MAXIMAL FUNCTIONS THROUGH FOURIER MULTIPLIERS

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Abstract

We study and prove endpoint bounds for derivatives of fractional maximal functions with either smooth convolution kernel or lacunary set of radii in dimensions $n = 2 + \delta , \delta \in \mathbb{N}$. We also show that the spherical fractional maximal function maps L^p into a first order Sobolev space in dimensions $n = 5 + \delta , \delta \in \mathbb{N}$.

عنوان البحث

كروية وحدود نقطة النهاية لمشتقات الدوال الاعظمية الكسرية عبر مضاعفات فورير

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المستخلص

درسنا وبرهنا حدود النقطة لمشتقات الدوال الاعظمية الكسرية مع كل من نواة الالتفاف الأملس أو الفئة الجوفاء من أنصاف الأقطار

في الابعاد $n = 2 + \delta , \delta \in \mathbb{N}$

أيضاً أوضحنا الدوال الأعظمية الكسرية الكروية L^p في فضاء رواسم سوبوليف ذو الرتبة الأولي في الأبعاد :

$n = 2 + \delta , \delta \in \mathbb{N}$.

1. Introduction

Define the fractional maximal function as

$$M_\alpha f(x) = sup_{t>0}\left|\frac{t^\alpha}{|B(x,t)|}\int_{B(x,t)} f\ dy\right|$$

For $\alpha \in [0, n)$. The study of its regularity properties was initiated in [22] by Kin-nunen and Saksman. They proved the pointwise inequality

$$|\nabla M_\alpha |f|(x)| \leq C M_{\alpha-1}|f|(x), \quad \alpha \geq 1 \tag{1.1}$$

with a constant C only depending on the dimension n and $\propto$. This inequality has two interesting consequences. First, $M_\propto$ maps $L^p(R^n)$ into a frst order Sobolev space. Second, as noted by Carneiro and Madrid [8], the pointwise bound together with the Gagliardo - Nirenberg - Sobolev inequality implies

$$\|\nabla M_\alpha f\|_{L^p} \leq C\|M_{\alpha-1} f\|_{L^p} \leq C\|f\|_{L^{n/(n-1)}} \leq C\|\nabla f\|_{L^1} \tag{1,2}$$

for $\alpha \geq 1$ and $P = n/(n-\alpha)$ When $\alpha \in (0,1)$, inequality (3.1) no longer helps, and the conclusion of (1.2) is an open problem. When M_α is replaced by its non-centred variant, the analogous result is due to Carneiro and Madrid [8] for $n = 1$ and Luiro and Madrid [28] for f radial and. For $n = 2$. other aspects of the regularity of fractional maximal functions, see e.g. [17, 18] and the references therein. The first result of David Beltran and Olli Saari is a smooth variant of the inequality (1.2) for $\alpha \in (0,1)$ and $n \geq 2$, [40]. Define the lacunary fractional maximal function as

$$M_\alpha^{lac} f(x) := sup_{k\in Z}\left|\frac{2^{\alpha k}}{|B(0,2^k)|}\int_{B(x,2^k)} f\ dy\right|.$$

For integrable φ and $t > 0$, let $\varphi_t(x) = t^{-n}\varphi(x/t)$. Assume, for simplicity, that φ is a positive Schwartz function and define the smooth fractional maximal function as

$$M_\alpha^\varphi f(x) = sup_{t>0} t^\alpha |\varphi_t * f(x)|.$$

The smoothness requirement can be substantially relaxed, see§§ 3.3.

Corollary(1.1)[40]: Let $f \in \dot{BV}(R^n)$ and suppose that $\alpha \in (0,1)$ and $n = 2 + \delta, \delta \in \mathbb{N}$. Then $M_\alpha \in \{M_\alpha^{lac}, M_\alpha^\varphi\}$,there exists a constant C only depending on dimension n, α and φ such that

$$\left\|\nabla M_{\alpha f}\right\|_{L^p(\mathbb{R}^{2+\delta})} \leq C|f|_{BV(\mathbb{R}^{2+\delta})}$$

$for\ p = 2 + \delta/(2 + \delta - \alpha)$

The proof of this corollary uses the g-function technique familiar from Stein's spherical maximal function theorem. The idea is to follow the scheme behind the short estimation (1.2). The Fourier transform is used to find a substitute for (1.1) at the level of Besov spaces, from which the conclusion then follows by a refined Gagliardo-Nirenberg-Sobolev type embedding theorem [10]. The last step requires $n > 1$ whereas the smoothness condition on the maximal operator is imposed by Fourier analysis. We stress that the right hand side of the conclusion is BV norm instead of the considerably larger homogeneous Hardy-Sobolev norm one might first expect. The detailed proof is given in §3, and all necessary definitions can be found in §2. To the best of our knowledge, Fourier transform techniques have not been exploited effectively in the study of endpoint regularity of maximal functions prior to this work.

The background of the question (1.2) goes back to Kinnunen's theorem [20, 21] asserting that the

Hardy-Littlewood maximal function is bounded in W1;p with $P > 1$. His result was later extended to $W^{1,1}$ in the form

$$\|\nabla Mf\|_{L^1(\mathbb{R}^{2+\delta})} \leq C\|\nabla f\|_{L^1(\mathbb{R}^{n+\delta})} \tag{1.3}$$

by Tanaka [38] when $n = 1$ and Luiro [27] when $n = 2 + \delta, \delta \in \mathbb{N}$[40] and f is radial. Here M is the noncentred Hardy {Littlewood maximal function. The same inequality for M_0 (centred maximal function) was established by Kurka [23] when $n = 1$, and the question is open in dimensions $n = 2 + \delta, \delta \in \mathbb{N}$[40]. Kurka's theorem can be seen as the limiting case $\alpha = 0$ of (1.2).

In connection to (1.3), maximal functions with smooth convolution kernels are better understood than the Hardy-Littlewood maximal function. Inequality (1.3) can be proved with sharp constant for many smooth kernels [7, 9] whereas the best constant for centred Hardy-Littlewood maximal function is not known (for the noncentred maximal function [2] as well as for certain non-tangential maximal functions [31] the constant is one). Similarly, a Hardy-Sobolev bound corresponding to (1.3) is known for smooth maximal functions in all dimensions [30] whereas the progress for the standard maximal function is limited to the case of radial functions [27]. Finally, there are metric measure spaces where Kinnunen's theorem does not hold but suitable smoother maximal functions satisfy a Sobolev bound [1]. Theorem 3.1 can be seen as a part of this line of research attempting to understand (1.2) and (1.3) first in the case of smooth maximal functions.

David Beltran and Olli Saari studies the regularity of the spherical fractional maximal function

$$S_\alpha f(x) := sup_{t>0}|t^\alpha \sigma_t * f(x)|, \tag{1.4}$$

where σ_t is the normalized surface measure of the sphere $\partial B(0,t)$. For $\alpha = 0$, one recovers the spherical maximal function of Stein [36] $(n \geq 3)$ and Bourgain [5] $n = 2$. For $\alpha > 0, L^p \to L^q$ bounds for this operator follow from the work of Schlag [33] $(n = 2)$ and Schlag and Sogge [34] $(n \geq 3)$. It is natural to ask if the fractional spherical maximal function has regularizing properties similar to (1.1) , [40]. The result of David Beltran and Olli Saari and our corollary in this direction is the following.

Corollary(1.2)[40]: Let $n = 5 + \delta, \delta \in \mathbb{N}, 5 + \delta/(3 + \delta) < \delta^2 + 4\delta + 5 \leq 2\delta^2 + 7\delta + 5 < \infty$ and

$$\alpha\left(\frac{\delta^2 + 4\delta + 5}{\delta + 3}\right):$$
$$= \begin{cases} \dfrac{\delta^4 + 12\delta^3 + 51\delta^2 + 96\delta + 70}{\delta^3 + 8\delta^2 + 21\delta + 20} & if \quad \dfrac{5+\delta}{3+\delta} < \dfrac{\delta^2 + 4\delta + 5}{\delta + 3} \leq \dfrac{\delta^2 + 10\delta + 26}{\delta^2 + 8\delta + 34} \\ \dfrac{\delta^2 + 7\delta + 12}{\delta^2 + 4\delta + 5} & if \quad \dfrac{\delta^2 + 10\delta + 26}{\delta^2 + 8\delta + 34} < \dfrac{\delta^2 + 4\delta + 5}{\delta + 3} \leq \delta + 4. \end{cases}$$

Assume that

$$\frac{\delta+3}{2\delta^2+7\delta+5} = \frac{\delta+3}{\delta^2+4\delta+5} - \frac{\alpha-1}{5+\delta}, \quad 1 \leq \alpha < \alpha(\frac{2\delta^2+7\delta+5}{\delta+3})$$

Then, for any $f \in L^{\left(\frac{2\delta^2+7\delta+5}{\delta+3}\right)}, S_\alpha f$ is weakly differentiable and

$$\|\nabla S_\alpha f\|_{L^{\left(\frac{2\delta^2+7\delta+5}{\delta+3}\right)}} \lesssim \|f\|_{L^{\left(\frac{\delta^2+4\delta+5}{\delta+3}\right)}}.$$

The proof of this corollary is also based on the use of the Fourier transform. When $q \geq 2$, we study $L^{\left(\frac{\delta^2+4\delta+5}{\delta+3}\right)} \to L^{\left(\frac{2\delta^2+7\delta+5}{\delta+3}\right)}$ [40] estimates for a maximal multiplier operator in analogy with the estimates in [33, 34, 25] for the spherical maximal function. Since Corollary(1.2) is a statement at the derivative level, the corresponding multiplier enjoys worse Fourier decay than $\widehat{\sigma}$. This forces us to study the behavior in $L^{\left(\frac{\delta^2+4\delta+5}{\delta+3}\right)}$ with large p more carefully than what is needed to understand $L^{\left(\frac{\delta^2+4\delta+5}{\delta+3}\right)}$ mapping properties of the spherical maximal function. We take advantage of the sharp local smoothing estimate for the wave equation in $L^{\delta+4}(\mathbb{R}^{\delta+5})$ which is available whenever $n = \delta + 5$, $\delta \in \mathbb{N}$ thanks to recent advances in decoupling theory (see [6, 14, 15, 24, 39] and [3, 19, 26, 29, 35] for more on decoupling and local smoothing estimates). We remark that results in $n = 4$ could be obtained upon further progress on local smoothing estimates.

2. Notation and Preliminaries

2.1. Notation. All function spaces are defined over, $\mathbb{R}^n$ and it is written, for instance L^2 for $L^2(\mathbb{R})$. The letter C denotes a generic constant whose value may vary from line to line. Its dependency on other parameters will be clear from the context. The notation $A \lesssim B$ is used if $A \le CB$ for such a constant C, and similarly $A \gtrsim B$ and $A \sim B$. The Fourier transform of a tempered distribution $f \in S$ is denoted by $\widehat{f}$ or $\mathcal{F}(f)$ and its inverse Fourier transform by $f^{-1}(f)$ or f^{V}; in particular for a Schwartz function$f \in S$,

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x.\xi} f(x).dx.$$

Given any multi-index $r \in N^n, \partial^r$ denotes

$$\partial^r f = \partial_{x_1}^{r_1} \dots \partial_{x_n}^{r_n} f.$$

For any $\alpha \in \mathbb{R}$, the notation $(-\nabla)^{\alpha/2}$is taken to denote the operator associated to the Fourier multiplier $|\xi|^\alpha$.

2.2. Besov spaces and Littlewood-Paley pieces. Given a smooth function $\psi \in C_c^\infty$ supported in$\{\xi \in \mathbb{R}^n : 2^{-1} < |\xi|\} < 2$ and such that

$$\sum_{j\in Z} \psi\left(2^{-j}\xi\right) = 1$$

for $\xi \neq 0$, let f_j denote the Littlewood-Paley piece of f at frequency 2^j, given by $\widehat{f_j} = \widehat{f}\,\psi\left(2^{-j}\xi\right)$ The Besov semi norm for $\dot{B}^s_{p,q}$for $s \in \mathbb{R}$ and $p, q \in [1, \infty]$ is defined as

$$\|f\|_{\dot{B}^s_{p,q}} = \left(\sum_{j\in Z} 2^{qjs} \|f_j\|_{L^p}^q\right)^{1/q};$$

the seminorms defined through different Littlewood-Paley functions ψ are comparable (see [4, Chapter 6] for further details).

2.3. BV space. A function f is said to have bounded variation, and denoted by $f \in \dot{BV}$, if its variation, defined by

$$|f|_{BV} \coloneqq \left\{\int_{\mathbb{R}^n} f\ diV(y);\ y \in C_c^1(\mathbb{R}^n, \mathbb{R}^n), \|y\|_\infty \le 1\right\},$$

is finite, where $g = \left(g_{1,} \dots\dots, g_n\right)$ and the L^∞ norm is defined by

$$\|g\|_\infty \left\| \left(\sum_{i=1}^n g_i^2\right)^{1/2} \right\|_{L^\infty}$$

Note that if f belongs to space $W^{1,1}$, integration by parts allows one to identify

$$|f|_{BV} = \int_{\mathbb{R}^n} |\nabla f|.$$

See [13, Chapter 5] for more.

2.4. Finite differences. Denote

$$D^h f(x) = \frac{f(x+h) - f(x)}{|h|}.$$

Recall (see e.g [12, Chapter 5, §5.8, Theorem 3.3.1]) that if there is a finite constant A such that

$$\|D^h f\|_{L^p} \leq A$$

for all $h \in \mathbb{R}^n$, then the weak derivatives of f exist and

$$\|\nabla\|_{L^p} \leq CA$$

for a constant C only depending on the dimension n. If S is a sublinear operator that commutes with translations, then

$$|D^h S f| \leq |S D^h f|.$$

In particular, if S is a maximal function and f is a positive function, this allows us to reduce the question about differentiability to bounded ness of a maximal multiplier for all Schwartz functions f.

3. Endpoint results

3.1. A model result. It is instructive to start first with a model case for corollary(1.1). This consists in the study of the single scale version of the (rough) fractional maximal function M_α, defined as

$$M_\alpha^* f = sup_{1 \leq t \leq 2} \left| \frac{1}{|B(x,t)|} \int_{B(x,t)} f(y) dy \right|.$$

Corollary(3. 1): Let $\alpha = 1 - \delta$, $p = 2 + \delta/(2\delta + 1)$ and $n = 2 + \delta$, $\delta \in \mathbb{N}$. Then there is a constant C only depending on dimension $n = 1 + \delta$, $\delta \in \mathbb{N}$and α such that for any $f \in \dot{B}^{\delta}_{2+\delta/(2\delta+1),1}$

$$\|M_{1-\delta}^* D^h f\|_{L^{2+\delta/(2\delta+1)}} \leq C \|f\|_{\dot{B}^{\delta}_{2+\delta/(2\delta+1),1}}$$

uniformly on $h \in \mathbb{R}^{2+\delta}$.

By the discussion in §§2.4, Corollary(3. 1) implies an $L^{2+\delta/(2\delta+1)}$ bound for the gradient of $M_{1-\delta}^*$. It will be shown in §§3.2 how the proof of the above estimate gives , Corollary(1. 1) for sightly smoother versions of the fractional maximal function, such as its lacunary version or maximal functions of convolution type with smooth kernels.

Proof. Write, for $f \in S$,

$$M_{1-\delta}^* (D^h f)(x) = sup_{1 \leq t \leq 2} \left| \mathcal{F}^{-1} \left((t|\xi|)^{1-\delta} \, \widehat{1_{B(0,1)}}(t\xi) \mathcal{F}\left(T_h (-\Delta) \right)^{\delta/2} f \right) \right|$$

where $T_h$is the operator defined by

$$\widehat{T_{hg}}(\xi) = \frac{e^{i\xi.h} - 1}{|\xi||h|} \hat{g}\,(\xi) =: a_h(\xi) \hat{g}(\xi). \tag{3.1}$$

Observe that T_h is a bounded operator on $L^{2+\delta/(2\delta+1)}$ uniformly in $h \in \mathbb{R}^{\delta+2}$ for all $1 < 2 + \delta/(2\delta + 1) < \infty$ by the Mikhlin-Hörmander multiplier theorem (see, for instance [11, Theorem 8.10]); it is clear that $|\partial^r a_h(\xi)| \lesssim |\xi|^{-|r|}$ for all multi-indexes $r \in \mathbb{N}_0^{\delta+2}$ with implicit constant independent of $h \in \mathbb{R}^{\delta+2}$. Thus, the operator T_h plays no role in determining the range of boundedness for $M_{1-\delta}^* D^h$.

Let $m(\xi) = |\xi|^{\alpha}\widehat{1_{B(0,1)}}(\xi)$ and $m_t(\xi) := m(t\xi)$ For all $t > 0$. For each $j \in \mathbb{Z}$, let $f_j = \psi_j * f$ denote the Littlewood-Paley piece of f around the frequency 2^j as in §§2.2. Assume momentarily that the following holds.

Corollary(3.2)[40]: Let $g \in S$. Then for $p = 2n/(n+2\delta)$ and $\alpha = n/2 - \delta$ $,\delta \in \mathbb{N}$

$$\left\|sup_{1<t<2}\left|\mathcal{F}^{-1}(m_t\hat{g}_j\right|\right\|_{L^{2n/(n+2\delta)}} \lesssim \left(2^{j(n/2-\delta\ )}1_{\{j\leq 0\}} + 1_{\{j>0\}}\right)\left\|g_j\right\|_{L^{2n/(n+2\delta)}.}$$

Then the proof may be concluded as follows. Decomposing the function f into frequency localised pieces $f_j$and applying Corollary(3.2)to the function $g = T_h(-\Delta)^{1/4(2-n+2\delta)}f$ one has

$$\left\|sup_{1<t<2}\left|\mathcal{F}^{-1}(m_t\hat{g}_j\right|\right\|_{L^{2n/(n+2\delta)}} \leq \sum_{j\in\mathbb{Z}}\left\|sup_{1<t<2}\left|\mathcal{F}^{-1}(m_t\hat{g}_j\right|\right\|_{L^{2n/(n+2\delta)}}$$

$$\lesssim \sum_{j\in\mathbb{Z}}\left(2^{j(n/2-\delta\ )}1_{\{j\leq 0\}} + 1_{\{j>0\}}\right)\left\|g_j\right\|_{L^{2n/(n+2\delta)}.}$$

$$\leq \sum_{j\in\mathbb{Z}} 2^{j1/2(2-n+2\delta)}\left\|f_j\right\|_{L^{2n/(n+2\delta)}} \sim \|f\|_{\dot{B}^{1/2(2-n+2\delta)}_{2n/(n+2\delta),1}}, \tag{3.2}$$

where the last step follows from the $L^{2n/(n+2\delta)}$ boundedness of $T_h$and Young's convolution inequality.

Remark (3.3): By Bernstein's inequality, $2^{j1/2(2-n+2\delta)}\left\|f_j\right\|_{L^{2n/(n+2\delta)}} \lesssim 2^j\left\|f_j\right\|_{L^1}$, so one may further bound $\|f\|_{\dot{B}^{1/2(2-n+2\delta)}_{2n/(n+2\delta),1}}, \lesssim \|f\|_{\dot{B}^1_{1,1}}$, in (3.2).

It remains to prove Corollary(3.2). This is done by interpolating an L^2 bound with an $L^1 - L^{1,\infty}$ bound as in the proof of the spherical maximal function theorem that can be found in the textbooks, see [37, Chapter XI, §3.3] or [16, Chapter 5.5]. Writing

$$\mathcal{F}^{-1}(m_t\hat{g}_j = t^{n/2-\delta}\ \mathcal{F}^{-1}\left(\widehat{1_{B(0,1)}}(t\xi)(|\xi|^{n/2-\delta}\ \hat{g}_j)\right),$$

it is clear that

$$sup_{1\leq t\leq 2}|\mathcal{F}^{-1}(m_t\hat{g}| \lesssim sup_{1\leq t\leq 2}\left|t^{-n}1_{B(0,t)} * ((-\Delta)^{1/2(n/2-\delta\ )}g)\right| \leq M((-\Delta)^{1/2(n/2-\delta\ )}g$$

where M is the Hardy-Littlewood maximal function. Bounds on M and Young's convolution inequality then imply

Proposition(3.4): Let g ∈ S. Then

$$\left\|sup_{1\leq t\leq 2}\left|\mathcal{F}^{-1}\left(m_t\hat{g}_j\right)\right|\right\|_{L^{1,\infty}} \lesssim 2^{j\alpha}\left\|g_j\right\|_{L^1}$$

The L^2 estimate follows by estimating the Fourier decay of m after an applicationof a Sobolev embedding. This is the part of the proof that allows to take advantageof better symbols m later in §§3.3 so we write the proof in detail.

Proposition (3.5): Let $g \in S$. Then

$$\left\|sup_{1\leq t\leq 2}\left|\mathcal{F}^{-1}\left(m_t\hat{g}_j\right)\right|\right\|_{L^2} \lesssim \left(2^{j\alpha}1_{\{j\leq 0\}} + 2^{j\alpha(-\frac{n}{2}+\alpha)}1_{\{j>0\}}\right)\left\|g_j\right\|_{L^{2}.}$$

Proof. Let $\tilde{m}(\xi) = \xi\ .\nabla m(\xi)$ and denote by $T_m$and $T_{\tilde{m}}$the operators associatedto the multipliers$m$ and $\tilde{m}$. By the fundamental theorem of calculus,

$$sup_{1\leq t\leq 2}\left|m_{mt}g_j\right| \leq \left|m_{mt}g_j\right| + 2\left(\int_1^2\left|T_{mt}g_j\right|\left|T_{\tilde{m}t}g_j\right|\frac{dt}{t}\right)^{1/2}$$

$$\leq |m_m g_j| + 2\left(\int_1^2 |T_{mt} g_j|\, 2\frac{dt}{t}\right)^{1/4} \left(\int_1^2 |T_{\tilde{m}t} g_j| 2\frac{dt}{t}\right)^{1/4} \quad (3,3)$$

Taking L^2-norm in the above expression, an application of the Cauchy-Schwarz inequality and Fubini's theorem reduces the problem to compute the L^∞norm of $m\psi_j$ and $\widetilde{m}\psi_j$.

Recall that $\widehat{1_{B(0,1)}}(\xi) = |2\pi\xi|^{-n/2} J_{n/2}(2\pi|\xi|$, where $J_{n/2}$denotes the Bessel function of order $n/2$, and

$$J_{n/2}(r) \lesssim r^{n/2} 1_{\{r\leq 1\}} + r^{-1/2} 1_{\{r>1\}};$$

see, for instance, [16, Appendix B] for further details. This immediately yields

$$\|m\psi_j\|_{L^\infty} \lesssim 2^{j\alpha} 1_{\{r\leq 0\}} + 2^{j(-\frac{n+1}{2}+\alpha)} 1_{\{j>0\}}. \quad (3.4)$$

Concerning $\widehat{m}$, the relation

$$\frac{d}{dr}\left[r^{-n/2} J_{n/2}(r)\right] = -r^{-n/2} J_{n/2+1}$$

and a similar analysis to the one carried above leads to

$$\|\widetilde{m}\psi_j\|_{L^\infty} \lesssim 2^{j\alpha} 1_{\{r\leq 0\}} + 2^{j(-\frac{n-1}{2}+\alpha)} 1_{\{j>0\}}.$$

Putting both estimates together in (3.3) concludes the proof corollary(3.2) now follows by interpolation, and the proof of the model case is complete.

3.2. **Extension to the full supremum**. From now on, we redefine m to be Fourier transform of an integrable function smoother than $1_{B(0,1)}$. Momentarily assume m satisfies

$$\left\|sup_{1\leq t\leq 2}\left|(m_t \hat{g}_j)^\vee\right|\right\|_{L^p} \lesssim \left(2^{j\alpha} 1_{\{j\leq 0\}} + 2^{-j\epsilon} 1_{\{j>0\}}\right) \|g_j\|_{L^p,} \quad (3,5)$$

which we next show to be enough to conclude a bound as in corollary(1.1). The proof of (3.5) is postponed to §§3.3.

Inequality (3.5) rescales as

$$\left\|sup_{2^{-k}\leq t\leq 2^{-k+1}}\left|(m_t \hat{g}_{j+k})^\vee\right|\right\|_{L^p} \lesssim \left(2^{j\alpha} 1_{\{j\leq 0\}} + 2^{-j\epsilon} 1_{\{j>0\}}\right) \|g_{j+k}\|_{L^p,} \quad (3.6)$$

In order to use this bound, break the full supremum over all possible scales and use the embedding $\ell^P \subseteq \ell^\infty$,

$$sup_{t<0}\left|(m_t \hat{g})^\vee\right| = sup_{k\in\mathbb{Z}}\ \ sup_{2^{-k}\leq t\leq 2^{-k+1}}\left|(m_t \hat{g})^\vee\right| \leq \left(\sum_{k\in\mathbb{Z}} sup_{2^{-k}\leq t\leq 2^{-k+1}}\left|(m_t \hat{g})^\vee\right|^p\right)^{1/p}$$

Taking L^pnorm and using (3.10), we see

$$\left\|sup_{t>0}\left|(m_t \hat{g})^\vee\right|\right\|_{L^p} \lesssim \sum_{j\in\mathbb{Z}} (2^{j\alpha} 1_{\{j\leq 0\}} + 2^{-j\epsilon} 1_{\{j>0\}}) \left(\sum_{k\in\mathbb{Z}} \|g_j + k\|_{L^p}^p\right)^{1/p}$$

Using the geometric decay to sum in $j \in \mathbb{Z}$ and recalling

$$\|g_{j+k}\|_{L^p} = \|(-\Delta)^{(1-\alpha)/2} f_{j+k}\|_{L^p} \lesssim 2^{(j+k)(1-\alpha)} \|f_{j+k}\|_{L^p}$$

we obtain

$$\left(\sum_{k\in\mathbb{Z}} \|g_j + k\|_{L^p}^p\right)^{1/p} \lesssim \|f\|_{\dot{B}^{1-}_{p,p}\alpha}$$

We then claim

$$\|f\|_{\dot{B}^{1-\alpha}_{p,p}} \lesssim |f|_{BV} \tag{3.7}$$

for $n > 1$ and $0 < \alpha < n/2$. This will follow from a Gagliardo-Nirenberg-Sobolev type inequality.

Corollary(3.6)[40]: ([10]). Assume $\gamma = 1 + \delta_1$ or $\gamma = 1 - \delta_1 - 1/n$, $\delta_1 > 0$ and let$(s;q)$ satisfy $(s-1)q'/n = \gamma - 1$ for some $q = 1 + \delta_2$, $\delta_2 > 0$ where $q' = 1 + \delta_2/\delta_2$. Then, for any $\theta = 1 - \delta_2$,

$$\|f\|_{\dot{B}^t_{(1+\delta_2)/\left(1+\delta_2-\delta_2{}^2\right),(1+\delta_2)/\left(1+\delta_2-\delta_2{}^2\right)}} \lesssim \|f\|^{(1-\delta_3)/(1+\delta_3)}_{\dot{B}^s_{1+\delta_2,1+\delta_2}} |f|^{(2\delta_3)/(1+\delta_3)}_{BV}$$

where $p = (1+\delta_2)/\left(1+\delta_2-\delta_2{}^2\right)$ and $t = \delta_2(s-1)+1$.

Indeed, taking $\gamma = 0, s = 1/2\,(1+\delta_3)$ and $\theta = 2\delta_3/1+\delta_3$, which are admissible for $n = 1 + \delta_3$ and $\alpha = 1/2\,(1-\delta_3)$, $\delta_3 > 0$ one has

$$\|f\|_{\dot{B}^{1/2(1+\delta_3)}_{(1+\delta_2)/\left(1+\delta_2-\delta_2{}^2\right),(1+\delta_2)/\left(1+\delta_2-\delta_2{}^2\right)}} \lesssim \|f\|^{(1-\delta_3)/(1+\delta_3)}_{\dot{B}^{1/2(1+\delta_3)}_{2,2}} |f|^{2\delta_3/1+\delta_3}_{BV}$$

Applying Bernstein's and Minkowski's inequalities as well as Littlewood-Paley the or y, we see

$$\|f\|_{\dot{B}^{1/2(1+\delta_3)}_{2,2}} \sim \left(\sum_{j\in\mathbb{Z}} 2^{2j(1/2(1+\delta_3))} \|f_j\|^2_{L^2}\right)^{1/2} \lesssim \left(\sum_{j\in\mathbb{Z}} 2^{2j(1/2(1+\delta_3))} 2^{2j1/2(\delta_3-1)} \|f_j\|^2_{L^{(1+\delta_3)/\delta_3}}\right)^{1/2}$$

$$= \left(\sum_{j\in\mathbb{Z}} \|f_j\|^2_{L^{(1+\delta_3)/\delta_3}}\right)^{1/2} \le \left\|\left(\sum_{j\in\mathbb{Z}} |f_j|^2\right)^{1/2}\right\|_{L^{(1+\delta_3)/\delta_3}} \sim \|f\|_{L^{(1+\delta_3)/\delta_3}}.$$

Inequality (3.7) then follows from the Gagliardo-Nirenberg-Sobolev inequality [13, Theorem 5.6.1. (i)], and we conclude

$$\|sup_{1\le t\le 2}|\mathcal{F}^{-1}(mt\hat{g})|\|_{L^{(1+\delta_2)/\left(1+\delta_2-\delta_2{}^2\right)}} \lesssim \|f\|^{(1-\delta_3)/(1+\delta_3)}_{L^{(1+\delta_3)/\delta_3}} |f|^{(2\delta_3)/(1+\delta_3)}_{BV} \lesssim |f|_{BV}.$$

Thus it suffices to verify (3.5). This is done separately in the cases when m comes from a smooth kernel and when the maximal function is lacunary.

3.3. Smooth kernel. Define the smooth fractional maximal function as follows. Let $\epsilon > 0$. Let φ be a positive function with radial L^1majorant such that $\hat{\varphi}(\xi) \lesssim \varphi\ |\xi|^{-n/2-\epsilon}$ whenever $|\xi| > 1$ and. For instance, any positive $Schwartz\ function\ or\ even$

$$\varphi(x) = (1-|x|^2)^{\epsilon}_{+}$$

with $\epsilon > 0$ will do (see Appendix B.5 in [16]). The subscript denotes the positive part as $f_+ = f.1_{\{f>0\}}$. Now we want to analyse M^{φ}_{α} as defined in the introduction. A repetition of the proof of Proposition (3.5) gives the L^2 bound

$$\left\|sup_{1\le t\le 2}\left|\mathcal{F}^{-1}(t|\xi|)^{\alpha}\hat{\varphi}(t\xi)\hat{g}_j\right|\right\|_{L^2} \lesssim \left(1_{\{j\le 0\}}2^{j\alpha} + 1_{\{j>0\}}2^{j(-\frac{n}{2}+\alpha-\epsilon)}\right)\|g_j\|_{L^2}.$$

The ϵ -decay gain in the above estimate continues to hold on $L^{n/(n-\alpha)}$, so the extra decay assumption (3.5) is satisfied for smooth convolution kernels. By §§3.2,Theorem (1.1) holds in this case.

3.4. Lacunary set of radii. Similarly, there is a gain in the L^2 estimate when we study the lacunary fractional maximal function. Now $m(\xi) = |\xi|^{\alpha}1_{\widehat{B(0,1)}}(\xi)$ and

$$c_n M^{lac}_{\alpha} f(x) = sup_{k\in\mathbb{Z}}\left|2^{k\alpha-nk}\int_{B(x,2^k)} f(y)dy\right| \le \left(\sum_{k\in\mathbb{Z}}\left|2^{k\alpha-nk}\int_{B(x,2^k)} f(y)dy\right|^p\right)^{1/p}$$

so that it suffices to use a bound for a single dilate (3.4) and replace the Proposition (3.5) by

$$\left\|\left|\mathcal{F}^{-1}\left(m\hat{g}_j\right)\right|\right\|_{L^2} \lesssim \left(2^{j\alpha}1_{\{j\le 0\}} + 2^{j\left(-\frac{n+1}{2}+\alpha\right)}1_{\{j>0\}}\right)\left\|g_j\right\|_{L^2,}$$

which has an extra $1/2$-decay compared to Proposition(3.5). After interpolation, this leads to an ϵ -decay gain in the $L^{n/(n-\alpha)}$estimate so that (3.5) (without supremum)and Theorem(1.1) for lacunary set of radii follow.

4. Proof of Corollary(1.2)

Recall the definition (1.4). By the characterization through finite differences described in §2, the sublinearity of $S_{1-\delta}$ and by density, it suffces to prove

$$\|S_{1-\delta}D^h f\|_{L^{(2\delta^2+7\delta+5)/(\delta+3)}} \lesssim \|f\|_{L^{(\delta^2+4\delta+5)/(\delta+3)}}$$

for all Schwartz functions f uniformly in$h \in \mathbb{R}^{(\delta+5)}$.

Observe that by means of Fourier transform,

$$S_{1-\delta}D^h(x) = sup_{t>0}\left|\mathcal{F}^{-1}\left(t^{1-\delta}\right)|\xi|\hat{\sigma}(t\xi)\mathcal{F}(T_h f)(x)\right)\right|, \tag{4.1}$$

where T_h is the Fourier multiplier operator (3.1). As described in §§3.1, $T_h$is bounded on $L^{1+\delta}$ for all $p = 1+\delta\,,\delta > 0$ uniformly in $h \in \mathbb{R}^{(\delta+5)}$ by the Mikhlin-Hörmander multiplier theorem, so it plays no role in determining the boundedness range for $S_{1-\delta}D^h f(x)$; for this reason, $T_h f$ is identified with f in the rest of this section.

4.1. The case $q = 2+\delta\,,\delta \ge 0$ It is enough to consider the single scale version of the maximal function in (4.1): suppose we can prove

$$\left\|sup_{1\le t>2}\left|\mathcal{F}^{-1}\left(t^{1-\delta}\right)|\xi|\hat{\sigma}(t\xi)\hat{f}_j\right)\right|\right\|_{L^{2+\delta}} \lesssim 2^{js_1}1_{\{j\le 0\}} + 2^{-js_2}1_{\{j>0\}}\left\|f_j\right\|_{L^{(\delta^2+4\delta+5)/(\delta+3)}} \tag{4.2}$$

For $s_1, s_2 > 0$. Then rescaling gives

$$\left\|sup_{2^{-k}\le t\le 2^{-k+1}}\left|\mathcal{F}^{-1}(t^{1-\delta})|\xi|\hat{\sigma}(t\xi)\hat{f}_{j+k}\right)\right|\right\|_{L^{2+\delta}} \lesssim 2^{js_1}1_{\{j\le 0\}} + 2^{-js_2}1_{\{j>0\}}\left\|f_{j+k}\right\|_{L^{(\delta^2+4\delta+5)/(\delta+3)}}$$

under the relation $\frac{1}{q} = \frac{1}{p} - \frac{\alpha-1}{n}$, and arguing as in §§3.2

$$\left\|\boldsymbol{sup_{t>0}}\left|\boldsymbol{\mathcal{F}^{-1}}\left(\boldsymbol{t^{1-\delta}}|\boldsymbol{\xi}|\boldsymbol{\hat{\sigma}}(\boldsymbol{t\xi})\boldsymbol{\hat{f}}\right)\right|\right\|_{L^{2+\delta}} \lesssim \sum_{j\in\mathbb{Z}}\left(\boldsymbol{2^{js_1}1_{\{j\le 0\}}} + \boldsymbol{2^{-js_2}1_{\{j>0\}}}\right)\left(\sum_{\boldsymbol{k}\in\mathbb{Z}}\left\|\boldsymbol{f_{j+k}}\right\|_{\boldsymbol{l^p}}^{\boldsymbol{2+\delta}}\right)^{\boldsymbol{1/2+\delta}} \lesssim \|\boldsymbol{f}\|_{\boldsymbol{L}^{(\delta^2+4\delta+5)/(\delta+3)}}.$$

where the last inequality follows from Minkowski's inequality $(q \ge p)$; controlling $l^{2+\delta}$norm by l^2 norm, and applying Littlewood-Paley theory to see the inner sum as $l^{(\delta^2+4\delta+5)/(\delta+3)}$norm of f. The sum in j converges as $s_1;\, s_2 > 0$. Hence it suffces to prove (4.2).

For low frequencies $j \le 0$, we can use domination by the Hardy-Littlewood maximal function, Young's convolution inequality and Bernstein's inequality to see

$$\left\|sup_{1\le t\le 2}\left|\mathcal{F}^{-1}(t^{1-\delta}|\xi|\hat{\sigma}\right|(t\xi)\hat{f}_j)\right\|L^{2+\delta} \lesssim \left\|M(-\triangle)^{1/2}f_j \parallel L^{2+\delta} \lesssim 2^{j(2-\delta)} \parallel f_j \parallel L^{(\delta^2+4\delta+5)/(\delta+3)}\right\|$$

Hence it suffices to prove (4.2) for j > 0.

4.2. A local smoothing estimate.

The Fourier transform of the spherical measure is

$$\hat{\sigma}\,(\xi) = 2\pi|\xi|^{-\frac{\delta+3}{2}} J_{\frac{\delta+3}{2}}(2\pi|\xi|) = \sum_{\pm} a_{\pm}(\,\xi) e^{\,\pm 2\pi i|\xi|}$$

Where the symbols $a_{\pm}$ are in the class $s^{-(\delta+4)/2}$, that is

$$\left|\partial_{\xi}^{\gamma} a_{\pm}(\xi)\right| \lesssim (1+|\xi|)^{-\frac{\delta+3}{2}} - |\gamma|$$

For all multi indices $\gamma \in \mathbb{N}_0^{\delta+5}$(c.f.[37,chapter VIII) . hence

$$\mathcal{F}^{-1}(\hat{\sigma}(t\xi)\hat{f} = \sum_{\pm} \int_{\mathbb{R}^{\delta+3}} e^{2\pi i(\xi.x \pm |\xi| t)} a_{\pm}(t\xi)\hat{f}(\xi),$$

so that the connection to half-wave propagator $e^{it\sqrt{-\triangle}} f(x) \coloneqq \int_{\mathbb{R}^{\delta+3}} e^{i.x.}\,\xi e^{it|\xi|\hat{f}(\xi)d\xi}$ is evident .we will quote the following result:

Corollary(4.1)[40]: (Consequence of [6]).*For* $n = 2+\delta\,, \delta \in \mathbb{N}\,, s \in \mathbb{R}$,

$$\left(\int_1^2 \left\| e^{it\sqrt{-\triangle}} f \right\|_{L^{(\delta^2+3\delta+6)/(1+\delta)}_{s-s_{(\delta^2+3\delta+6)/(1+\delta)}+\theta}(\mathbb{R}^{2+\delta})}^{(\delta^2+3\delta+6)/(1+\delta)} dt\right)^{1/p} \lesssim \|f\|_{L_s^{(\delta^2+3\delta+6)/(1+\delta)}(\mathbb{R}^{2+\delta})}$$

holds for $\theta = (1-\delta^3-3\delta^2-5\delta)/(\delta^2+3\delta+6)$, $\delta \in \mathbb{N}$, and $s_{(\delta^2+3\delta+6)/(1+\delta)} = (1+\delta)\left(\frac{1}{2} - \frac{(\delta^2+3\delta+6)}{(1+\delta)}\right)$ whenever $p = \frac{(\delta^2+3\delta+6)}{(1+\delta)}$.

This can be found as Corollary(1.3) (i) in [14] knowing that the conjectured value of p_d in Table 1 of that paper has later been verified by [6].

Proposition (4.2): Let g be a Schwartz function and $j > 0$. For any $\epsilon > 0$

$$\left\| sup_{1 \le t \le 2} \left|\sigma_t * g_j\right| \right\|_{L^{n-1}} \lesssim 2^{j(\epsilon-1)} \left\|g_j\right\|_{L^{n-1}}$$

Proof. For $j > 0$ and a smooth bump x around [1, 2], we have

$$\left\| sup_{1 \le t \le 2} \left|\sigma_t * g_j\right| \right\|_{L^{n-1}} \lesssim \left\| (1+\sqrt{-\partial_t^2})^r x.\sigma_t * g_j \right\|_{L^{n-1}(\mathbb{R}^n+1)}$$

$$\lesssim 2^{j(r+s_{p-\theta-}\frac{n-1}{2}+\epsilon)} \left\|g_j\right\|_{L^{n-1}(\mathbb{R}^n)}$$

where we used Sobolev embedding with $r > 1/(n-1)$, Corollary(4.1) with $p = n-1$ as well as Young's convolution inequality. Simplifying the exponent in accordance with Corollary(4.1), we obtain the claim.

4.3. $L^p \to L^q$ estimates**.** To finish the proof of (4.2), we prove $L^p \to L^q$ estimates following the interpolation scheme of Lee [25] enhanced with the sharp local smoothing estimate. Denote

$$S_j^* f(x) \coloneqq \; sup_{1 \le t \le 2} \left|\mathcal{F}^{-1}\left(\hat{\sigma}(t\xi)|\xi|\widehat{f_j}(\xi)\right)(x)\right|$$

where $\widehat{f_j} = \hat{f}\psi_j$ still stands for Fourier localization at the level of a Littlewood-Paley piece of frequency 2^j .

Proposition (4.3): Let P be the open convex polygon with vertices

$$A = \left(\frac{n-2}{n} - \frac{2}{n}\right) \quad , \quad B = \left(\frac{n^2-2n-1}{n^2+1}\,, \frac{2(n-1)}{n^2+1}\right)$$

$$C=\left(\frac{1}{n-1},\frac{1}{n-1}\right)\quad,\qquad D=\left(\frac{n-2}{n},\frac{n-2}{n}\right)$$

Then

$$\left\|S_j^*f\right\|_{L^q}\lesssim 2^{-\varepsilon j}\left\|f_j\right\|_{L^p}$$

for some $\varepsilon > 0$ and all $j > 0$ provided that $(1/p\ ,\ 1/q)\in P$.

Proof. Since supp $\hat{\sigma}.\psi_j(t\xi)\subset\left\{\left|\xi\backsim 2^j\right|\right\}$, we can assume that b $\hat{f}$ is supported in anannulus around $|\xi|=2^j$. We use the following bounds:

$$\left\|S_j^*f\right\|_{L^1}\lesssim 2^{2j}\left\|f_j\right\|_{L^1}$$

$$\left\|S_j^*f\right\|_{L^\infty}\lesssim 2^{2j}\left\|f_j\right\|_{L^1}$$

$$\left\|S_j^*f\right\|_{L^{n-1}}\lesssim 2^{2}\left\|f_j\right\|_{L^{n-1}}\ \text{ for all }\delta > 0 \tag{4.3}$$

$$\left\|S_j^*f\right\|_{L^2}\lesssim 2^{\frac{n-4}{2}j}\left\|f_j\right\|_{L^2}$$

$$\left\|S_j^*f\right\|_{L^{\frac{2(n+1)}{n-1}}}\lesssim 2^{-j\frac{n^2-4n-3}{2n+2}}\left\|f_j\right\|_{L^2}.$$

To verify (4.3), use Proposition(4.2) as well as Young's convolution inequality to obtain

$$\left\|S_j^*f\right\|_{L^{n-1}}\lesssim 2^{-j(1-\delta)}\left\|(-\Delta)^{1/2}f\right\|_{L^{n-1}}\lesssim 2^{j\delta}\left\|f_j\right\|_{L^{n-1}}.$$

The other inequalities follow similarly, that is, by borrowing the corresponding bounds for the spherical maximal function (inequalities (1.7) - (1.10) in [25]), and applying Young's convolution inequality. Interpolating the bounds above, we obtain the claimed proposition

For each $p > 1$, we want to find the values of α such that $(1/p\ ;\ 1/q)\in P$ when $(\alpha-1)/n={}^1/_p-{}^1/_q$ and $q\ \geq\ 2$. When $q\ \geq\ 2$ is assumed, this happens when

$$\frac{n}{n-2}<p\leq\frac{n^2+1}{n^2-2n-1}\ ,\qquad \alpha<\frac{n^2-2n-1}{n-1}-\frac{2n}{p(n-1)}$$

or

$$\frac{n^2+1}{n^2-2n-1}<p\leq n-1,\ \ \alpha\frac{n-1}{p}.$$

This concludes the proof for the case $q\ \geq\ 2$. Notice that the restriction $q\ \geq\ 2$ is not dictated by validity of $L^p\ \to L^q$ estimates but it was required in order to upgrade the single scale bounds to bounds for the full maximal operator in §§4.1.

4.4. The case $q\ \leq\ 2$. Next we remove the assumption $q\ >\ 2$. Let

$$T^*f(x)=\ sup_{t>0}\left|\mathcal{F}^{-1}\left((t|\xi|)^\alpha\hat{\sigma}(t\xi)\hat{f}(\xi)\right)(x)\right|.$$

The operator S_α in (4.1) can be written

$$S_\alpha = T^*I_{\alpha-1}T_hf$$

where $\widehat{I_{\alpha-1}f}=|\xi|^{1-\alpha}\hat{f}$ is the Riesz potential of order $\alpha-1$ and T_h are as in (3.1). As discussed in §§3.1, T_h are bounded in L^p for all $p\ >\ 1$. Also, by the Hardy-Littlewood-Sobolev inequality $I_\alpha-1$

is bounded $L^p \to L^q$; for p, q obeying $\frac{\alpha-1}{n} = \frac{1}{p} - \frac{1}{q}$. Therefore, it is enough to analyse the operator T^*.

Let $m(\xi) = |\xi|^\alpha \hat{\sigma}(\xi)$ and take a Littlewood-Paley function ψ (as in §2). We define $m_1 = \sum_{j>0} \psi_j\, m$ and $m_0 = \sum_{j\le 0} \psi_j m$ Take T_j^* to be as T^* but m replaced by m_j . Then

$$T^* f \le T_0^* + T_1^* f.$$

We first bound T_0^* . A straightforward computation shows that m0 is bounded and for any multi-index $\beta \in \mathbb{N}^n$ with $|\beta| = k,\ k \le n + 1$

$$\left|\partial_\xi^\beta\, m_0(\xi)\right| \lesssim |\xi|^{\alpha-k}$$

so that

$$\|(1 + |.|)^{n+1} \mathcal{F}^{-1}(m_0)\|_{L^\infty} \lesssim 1$$

(because $\alpha > 1$). Consequently

$$T_0^* f \lesssim Mf$$

and boundedness in any L^p with $p > 1$ follows from that of the Hardy-Littlewood maximal function.

To bound T_1^* , we use a part of Theorem B from [32]:

Theorem (4.4): (Rubio de Francia [32]). Let m be a function in $C^{s+1}(R^n)$ for some integer $s > n/2$ such that $|D^\alpha m(\xi)| \lesssim |\xi|^{-a}$ for all $|\alpha| \le s + 1$. Suppose also that $a > \frac{1}{2}$: Then the maximal multiplier operator $T^* f := sup_{t>0}\left|\mathcal{F}^{-1}(m(t\cdot)\hat{f}\right|$ is bounded in L^r, for

$$\frac{2n}{2n+2n-1} < r \le 2$$

Since $\sum_{j<0} \psi_j m$ is smooth and satis $|D^\alpha m(\xi)| \lesssim |\xi|^{-a}$, for all $|\alpha| \le s + 1$

with $a = \frac{n-1}{2} - \alpha$, we can apply the theorem to conclude the proof whenever

$$\frac{2n}{2n-2-2\alpha} < q \le 2\,, \quad a > \frac{1}{2}$$

which is equivalent to $p > \frac{n}{n-2}$ and $\alpha < \frac{n-2}{2}$. However, given $p > \frac{n}{n-2}$, the condition $\alpha > \frac{n}{n-2}$ is automatically satisfied whenever $q \le 2$. Hence $\alpha < \alpha(p)$ is an active constraint only when $q > 2$.

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