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RESEARCH ARTICLE

ON SOME COMMON FIXED POINT THEOREMS IN A CONE *S*- METRIC SPACE USING WEAKLY COMPATIBLE SELF MAPS WITH (E-A) PROPERTY

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Abstract

In this paper, we prove a common fixed point theorem for weakly compatible self maps with (E-A)property of a cone S- metric space, An illustrative application is given to support our result. These results generalize some well-known recent results.

1 Introduction

Banachâ€™s contraction principle in metric spaces is one of the most important feedback in the theory of fixed points and non-linear analysis in general. From 1922, when Stefan Banach formulated the notion of contraction and proved the famous theorem, scientists around the world are publishing new results that are connected either to establish a generalization of metric space or to get a improvement of contractive condition. .G.Jungck [8],[9] first introduced compatible mappings and later Jungck and Rhoades [10] introduced the notion of weakly compatible mappings as a generalization of weakly commuting mappings given by Sessa [12]. Recently, Aamri and Moutawakil [7] introduced the concept of (E.A) property. One of the generalization of metric spaces is given in the paper of Sedghi et al. In 2012 [1]. They introduced a notion of S-metric spaces and give some of their properties. For more details regarding this spaces we refer [1],[2]. For the sake of transparency, we list the basic properties of an S-metric spaces that will be used later. In 2017[11] Krishnakumar and Dhamodharan introduced the concept of generalised S-metric space of cone S-metric space and prove some fixed point theorems of contractive mappings. Huang and Zhang [13] introduced the concept of cone metric space by replacing the set of real numbers by a general Banach space E which is partially ordered with respect to a cone $P \subset E$. Dhamodharan and Krishnakumar [15] also further extended S-metric space to cone S-metric space. In our paper, we prove a common fixed point theorem for weakly compatible maps (E-A) property in a Cone S metric space. Our paper extends and improve several previous results.

2 Preliminaries

In [13], let E be a Banach space. A subset P of E is called a cone if and only if:

1. P is closed, nonempty and $P \neq 0$
2. $ax + by \in P$ for all $x, y \in P$ and nonnegative real numbers a, b
3. $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We will write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while x, y will stand for $y - x \in intP$, where $intP$ denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that $0 \leq x \leq y$ implies $\| x \| \leq K \| y \|$ for all $x, y \in E$. The least positive number satisfying the above is called the normal constant.

The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is sequence such that $x_1 \leq x_2 \leq \cdots \leq x_n \cdots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\| x_n - x \| \to 0$ as $n \to 0$. Equivalently the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone. Suppose E is a Banach space, P is a cone in E with $intP \neq 0$ and $\leq$ is partial ordering with respect to P.

Definition 2.1 *[11] Suppose that E is a real Banach space, then P is a cone in E with $intP \neq \emptyset$, and $\leq$ is partial ordering with respect to P. Let X be a non-empty set, a function $d: X \times X \times X \to E$ is called a cone S metric on X if it satisfies the following conditions with*

1. $S(u, v, z) \geq 0$
2. $S(u, v, z) = 0$ if and only if $u = v = z$.
3. $S(u, v, z) \leq S(u, u, a) + S(v, v, a) + S(z, z, a)$

Then the function S is called an cone S-metric on X and the pair (X, S) is called an cone S-metric space simply CSMS.

Example 2.2 *[11] Let $E = R^2$, $P = \{(x, y): x, y \geq 0\}$, $X = R$ and $d: X \times X \times X \to E$ such that then $S(x, y, z) = (d(x, z) + d(y, z), \alpha(d(x, z) + d(y, z)))$, $(\alpha > 0)$ is an cone S- metric on X.*

Example 2.3 *[11] Let (X, d) be a cone metric space. Define $S: X \times X \times X \to E$ by $S(x, y, z) = d(x, z) + d(y, z) + d(z, x)$ for every $x, y, z \in X$*

Example 2.4 *[11] Let $E = R^3$, $P = \{(x, y, z): x, y, z \geq 0\}$, $X = R$ and $d: X \times X \times X \to E$ such that*

$$S(u,u,u) = (0,0,0) = S(v,v,v)$$
$$S(u,v,v) = (0,1,1) = S(v,u,v) = S(u,u,v)$$
$$S(v,u,u) = (0,1,0) = S(u,v,u) = S(u,v,u)$$

Here (x, S) is cone S metric space but not a G-cone metric space since $S(u, u, v) \neq S(u, v, v)$

Lemma 2.5 *[11] Let (X, S) be an cone S-metric space . Then we have $S(u, u, v) = S(v, v, u)$*

Definition 2.6 *[11] Let (X,S) be an cone S-metric space .*

1. A sequence $\{u_n\}$ in X converges to u if and only if $S(u_n,u_n,u) \to 0$ as $n \to \infty$. That is, there exists $n_0 \in N$ such that for all $n \geq n_0, S(u_n,u_n,u) \ll c$ for each $c \in E$, $0 \ll c$. We denote this by $\lim_{n\to\infty} u_n = u$ or $\lim_{n\to\infty} S(u_n,u_n,u) = 0$.

2. A sequence $\{u_n\}$ in X is called a Cauchy sequence if $S(u_n,u_n,u_m) \to 0$ as $n,m \to \infty$. That is, there exists $n_0 \in N$ such that for all $n,m \geq n_0$, $S(u_n,u_n,u_m) \ll c$ for each $c \in E$, $0 \ll c$.

3. The cone S-metric space (X,S) is called complete if every Cauchy sequence is convergent.

In the following lemma we see the relationship between a cone metric and an cone S-metric. Now, we give the definition of weakly compatible of cone S-metric space

Definition 2.7 *Let (A,B) be two self maps mappings of cone S a metric space (X,S), The pair (A,B) is said to be weakly compatible, if $S(ABx,ABx,BAx) \ll 0$, whenever $S(Ax,Ax,Bx) \ll 0$. That is the mappings A and B are said to be weakly compatible if they commute at their coincident points.*

Definition 2.8 *Let (A,B) be two self maps mappings of a cone S metric space (X,S). We say that A and B satisfies the property (E.A) if there exists a sequence x_n in X such that $lim_{n\to\infty}Bx_n = lim_{n\to\infty}Ax_n = s$ for $s \in X$.*

Example 2.9 *let $X = P \cup \{0\}$. Define $f,g: X \to X$ by $Ax = \frac{x}{5}$ and $Bx = \frac{3x}{5}$ for all $x \in X$, Consider the sequence $x_n = \frac{1}{n^2}$. Clearly $lim_{n\to\infty}Bx_n = lim_{n\to\infty}Ax_n = 0$*

Then A and B satisfy (E.A) property.

Example 2.10 *Let $X = P$ where $P \subseteq [3,\infty) \subset R$. Since $A,B: X \to X$ by $Ax = x+2$ and $Bx = 2x+2$ for all $x \in X$, suppose that property (E.A) holds, then there exist a sequence x_n in X satisfying $lim_{n\to\infty}Ax_n = lim_{n\to\infty}Bx_n = r$ for some $r \in X$, then $lim_{n\to\infty}x_n = r-2$,*

$\lim_{n\to\infty}x_n = \frac{r-2}{2}$, then $r = 2$ contradiction since $2 \notin X$

In (2002) M. Aamri and D. El Moutawakil , prove The following theorem.

Theorem 2.11 *[7] Let S and T be weakly compatible mappings of a metric space (X,d) such that: (i) T and S satisfy (E.A) - property, (ii) $d(Tx,Ty) < max\{d(Sx,Sy), \frac{1}{2}[d(Sx,Tx)+d(Sy,Ty)], \frac{1}{2}[d(Sx,Ty)+d(Sy,Tx)]\}$,*

for all $x \neq y \in X$ $(iii) T(X) \subset S(X)$. If $S(X)$ or $T(X)$ is a complete subspace of X, then T and S have a unique common fixed point.

Definition 2.12 *[14] Suppose that P is a normal cone in E. $a,b \in E$ and $a < b$. we define*

$$[a,b] = \{x \in E: x = tb + (1-t)a, \text{forsome} t \in [0,1]\} \quad (2.1)$$

Respectively.

Definition 2.13 *Suppose that P is a normal cone in E. $\zeta: [a,b] \to P$ is called an integrable function on $[a,b]$ with respect to cone P or to simplicity, Cone integrable function, if and only if for all partition Q of $[a,b]$, $\lim_{n\to\infty} L_n^{con}(\zeta,Q) = S^{con} = \lim_{n\to\infty} U_n^{con}(\zeta,Q)$, where S^{con} must be unique. We show the common value S^{con} by $\int_a^b \zeta(x)d_p(x)$ tosimplicity $\int_a^b \zeta d_p$*

Definition 2.14 *The function $\zeta: P \to E$ is called sub-additive cone integrable function if and only if for all $a,b \in P$,*

$$\int_0^{a+b} \zeta d_p \leq \int_0^a \zeta d_p + \int_0^b \zeta d_p$$

Example 2.15 *Let* $E = X = R, d(x,y) = |x-y|, P = (0,\infty)$, *and* $\zeta(t) = \frac{1}{(t+1)}$ *for all* $t > 0$. *Then for all* $a, b \in P$,

$\int_0^{a+b} \frac{dt}{(t+1)} = \ln(a+b+1)$, $\int_0^{a} \frac{dt}{(t+1)} = \ln(a+1)$, $\int_0^{b} \frac{dt}{(t+1)} = \ln(b+1)$ Since $ab \geq 0$, then $a + b + 1 \leq a + b + 1 + ab = (a+1)(b+1)$. Therefore

$$\ln(a+b+1) \leq \ln(a+1) \leq \ln(b+1)$$

This shows that ζ is an example of sub-additive cone integrable function.

3 main result

Theorem 3.1 *Let* A, B, H *and* P *be a self-maps of a cone* S*-metric space* (X, S) *and* P *is a normal cone satisfying the following conditions:* (1) $A(X) \subseteq H(X)$ *,and* $B(X) \subseteq P(X)$. (2) *one of* $A(X), B(X), H(X)$ *and* $P(X)$ *is closed subset of* X. (3) $S(Ax,Ax,By) \leq max\{S(Px,Px,Hy), \frac{1}{2}[S(Ax,Ax,Px) + S(Hy,Hy,By)], \frac{1}{2}[S(Px,Px,By) + S(Ay,Ay,Hx)]\}$, (4) *The pairs* (A,P) *and* (B,H) *are weakly compatible.* (5) *The pairs* (A,P) (B,H) *satisfies the property (E-A).Then* A, B, H *and* P *have a unique common fixed point in* X.

Proof. We, first prove the existence of a common fixed point in one of the two cases of the condition (5)and the other case follows similarly with appropriate changes, here we prove the case (B,H) satisfies the property (E-A), then there exist a sequence $\{x_n\}$ in X such that,

$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Hx_n = w \text{forsome} w \in X$$

since, $B(X) \subseteq P(X)$, there exists a sequence $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Py_n = w \text{forsome} w \in X$$

Now, we prove $\lim_{n\to\infty} Ax_n = w$, let $Ay_n = r$ and if $r \neq w$, then $S(r,r,w) \neq 0$ from (3), we have

$$S(Ax_n,Ax_n,Bx_n) < \max\{S(Py_n,Py_n,Hx_n)$$
$$,\frac{1}{2}[S(Ay_n,Ay_n,Py_n) + S(Hx_n,Hx_n,Bx_n)]$$
$$,\frac{1}{2}[S(Py_n,Py_n,Bx_n) + S(Ay_n,Ay_n,Hx_n)]\}$$

let $n \to \infty$

$$S(r,r,w) < \max\{S(w,w,w)$$
$$,\frac{1}{2}[S(r,r,w) + S(w,w,w)]$$
$$,\frac{1}{2}[S(w,w,w) + S(r,r,w)]\}$$

$$S(r,r,w) < \max\{0, \frac{1}{2}[S(r,r,w) + 0], \frac{1}{2}[0 + S(r,r,w)]\}$$

$$S(r,r,w) < S(r,r,w)$$

which is contradiction since, $r = w$. Hence we have $\lim_{n\to\infty} Ay_n = w$

Suppose $P(X)$ is closed subset of X, then there exist $q \in X$ such that $Pq = w$ therefore, we have

$$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Hx_n = \lim_{n\to\infty} Px_n = w = Pq, q \in X$$

Now, from (3) we have

$$S(Aq,Aq,Bx_n) < \max\{S(Pq,Pq,Hx_n)$$
$$,\frac{1}{2}[S(Aq,Aq,Pq) + S(Hx_n,Hx_n,Bx_n)]$$
$$,\frac{1}{2}[S(Pq,Pq,Bx_n) + S(Aq,Aq,Hx_n)]\}$$

$$\lim_{n\to\infty} \parallel S(Aq,Aq,Bx_n) \parallel < K \parallel \max\{S(Pq,Pq,Hx_n)$$
$$,\frac{1}{2}[S(Aq,Aq,Pq) + S(Hx_n,Hx_n,Bx_n)]$$
$$,\frac{1}{2}[S(Pq,Pq,Bx_n) + S(Aq,Aq,Hx_n)]\} \parallel$$

Thus $S(Aq,Aq,Bx_n) = 0$ which is implies that $S(Aq,Aq,Bx_n) \ll 0$. Thus $Aq = w$ therefore, we have, $Aq = Pq = w$

since, $A(X) \subseteq H(X)$, there exists a sequence $v \in X$ such that $Aq = Hl = w$

Now, we claim that $Bl = w$, if $Bl \neq w$, then $S(w,w,Bl) > 0$ from (3), we have

$$S(Aq,Aq,Bl) < \max\{S(Pq,Pq,Hl)$$
$$,\frac{1}{2}[S(Aq,Aq,Pq) + S(Hl,Hl,Bl)]$$
$$,\frac{1}{2}[S(Pq,Pq,Bl) + S(Aq,Aq,Hl)]\}$$

$$S(w,w,Bl) < \max\{S(w,w,w),$$
$$,\frac{1}{2}[S(w,w,w) + S(w,w,Bl)]$$
$$,\frac{1}{2}[S(w,w,Bl) + S(w,w,w)]\}$$

$$S(q,q,Bl) < \max\{0,\frac{1}{2}[0 + S(w,w,Bl)],\frac{1}{2}[S(w,w,Bl) + 0]\}$$

which is contradiction since, $Bl = w$.

therefore, we have, thus $Hl = Bl = w$

since, the pair (A,P) is weakly compatible then, $APq = PAq \Rightarrow Aw = Pw$.

We now show that $Aw = w$ if $Aw \neq w$ then $S(Aw,w,w) > 0$ from (3) we have

$$S(Aw,Aw,Bl) < \max\{S(Pw,Pw,Hl),$$
$$\frac{1}{2}[S(Aw,Aw,Pw) + S(Hl,Hl,Bl)],$$
$$\frac{1}{2}[S(Pw,Pw,Bl) + S(Aw,Aw,Hl)]\}$$

$$S(Aw,Aw,w) < \max\{S(Aw,Aw,w),$$
$$\frac{1}{2}[S(Aw,Aw,Aw) + S(w,w,w)],$$
$$\frac{1}{2}[S(w,w,w) + S(Aw,Aw,w)]\}$$

$$S(Aw,Aw,w) < \max\{S(Aw,Aw,w),$$
$$\frac{1}{2}[S(Aw,Aw,Aw) + 0],$$
$$\frac{1}{2}[0 + S(Aw,Aw,w)]\}$$

which is contraction then $Aw = Pw = w$, now we prove w is common fixed point of A and P

Since, the pair (B,H) is weakly compatible then, $BHl = HBl \Rightarrow Bw = Hw$, we prove $Bl = l$

$$S(Aw,Aw,Bw) < \max\{S(Pw,Pw,Hw),$$
$$\frac{1}{2}[S(Aw,Aw,Pw) + S(Hw,Hw,Bw)],$$
$$\frac{1}{2}[S(Pw,Pw,Bw) + S(Aw,Aw,Hw)]\}$$

$$S(w,w,Bw) < \max\{S(w,w,Bw),$$
$$\frac{1}{2}[S(w,w,w) + S(Bw,Bw,Bw)],$$
$$\frac{1}{2}[S(w,w,Bw) + S(w,w,Bw)]\}$$

$$S(w,w,Bw) < \max\{S(w,w,Bw),0,\frac{1}{2}[S(w,w,Bw) + S(w,w,Bw)]\}$$

which is contradiction then $Bw = Hw = w$, Similarly, we can prove w is common fixed point of B and H.

Now, we prove the uniqueness of common fixed point of A, B, H and P, if $w \neq u, u \in X$, then $S(w,w,u) > 0$ from (3),

$$S(w,w,u) < \max\{S(Pw,Pw,Hw),$$
$$\frac{1}{2}[S(Aw,Aw,Pw) + S(Hw,Hw,Bw)],$$
$$\frac{1}{2}[S(Pw,Pw,Bw) + S(Aw,Aw,Hw)]\}$$

$$S(w,w,u) < \max\{S(w,w,u),$$
$$\frac{1}{2}[S(w,w,w) + S(u,u,u)],$$
$$\frac{1}{2}[S(w,w,u) + S(w,w,u)]\}$$

$$S(w,w,u) < \max\{S(w,w,u),0,0\}$$

which is contradiction then, then w is common fixed point.

Corollary 3.2 *Let A, B, H and P be a self-maps of a cone S-metric space (X, S) and P is a normal cone satisfying the following conditions:* (1) $A(X) \subseteq P(X)$ *,and* $B(X) \subseteq P(X)$.

(2) one of $A(X), P(X), H(X)$, and $P(X)$ is closed subset of X.

(3) $S(Ax, Ax, By) < \max\{S(Px, Px, Py), \frac{1}{2}[S(Ax, Ax, Px) + S(Py, Py, Py)], \frac{1}{2}[S(Px, Px, By) + S(Ax, Ax, Py)]\}$,

(4) The pairs (A, P) and (B, P) are weakly compatible.

(5) The pairs (A, P) (B, P) satisfies the property (E-A).Then A, B, H and P have a unique common fixed point in X.

Corollary 3.3 *Let A, B, H and P be a self-maps of a cone S-metric space (X, S) and P is a normal cone satisfying the following conditions:* (1) $A(X) \subseteq P(X)$.

(2) one of $A(X), P(X), H(X)$, and $P(X)$ is closed subset of X.

(3) $S(Ax, Ax, Ay) < \max\{S(Px, Px, Py), \frac{1}{2}[S(Ax, Ax, Px) + S(Py, Py, Ay)], \frac{1}{2}[S(Px, Px, Ay) + S(Ax, Ax, Py)]\}$,

(4) The pairs (A, P) is weakly compatible.

(5) The pairs (A, P) satisfies the property (E-A).Then A, B, H and P have a unique common fixed point in X.

4 Some integral type contraction

In this we discussed some integral type contraction is satisfying (E.A) property.

Theorem 4.1 *Let A, B, H and P be a self-maps of a cone S-metric space (X, S) and P is a normal cone satisfying the following conditions:* (1) $A(X) \subseteq H(X)$ *,and* $B(X) \subseteq P(X)$. (2) *one of* $A(X), B(X), H(X)$ *and* $P(X)$ *is closed subset of* X. (3) $\int_0^{S(Ax,Ax,By)} \zeta d_p \leq$
$max\{\int_0^{S(Px,Px,Hy)} \zeta d_p, \int_0^{\frac{1}{2}[S(Ax,Ax,Px)+S(Hy,Hy,By)]} \zeta d_p, \int_0^{\frac{1}{2}[S(Px,Px,By)+S(Ay,Ay,Hx)]} \zeta d_p\}$,

where the function $\zeta: P \to P$ be defined as for each $\epsilon > 0, \int_0^{\epsilon} \zeta d_p > 0$ (4) The pairs (A, P) and (B, H) are weakly compatible. (5) The pairs (A, P) (B, H) satisfies the property (E-A).Then A, B, H, P have a unique common fixed point in X.

Proof. We, first prove the existence of a common fixed point in one of the two cases of the condition (5)and the other case follows similarly with appropriate changes, here we prove the case (B, H) satisfies the property (E-A), then there exist a sequence $\{x_n\}$ in X such that,

$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Hx_n = w \text{forsome} w \in X$$

since, $B(X) \subseteq P(X)$, there exists a sequence $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Py_n = w \text{forsome} w \in X$$

Now, we prove $\lim_{n\to\infty} Ax_n = w$, let $Ay_n = r$ and if $r \neq w$, then $S(r, r, w) \neq 0$ from (3), we have

$$\int_0^{S(Ax_n,Ax_n,Bx_n)} \zeta d_p < \max\{\int_0^{S(Py_n,Py_n,Hx_n)} \zeta d_p$$
$$, \int_0^{\frac{1}{2}[S(Ay_n,Ay_n,Py_n)+S(Hx_n,Hx_n,Bx_n)]} \zeta d_p$$
$$, \int_0^{\frac{1}{2}[S(Py_n,Py_n,Bx_n)+S(Ay_n,Ay_n,Hx_n)]} \zeta d_p\}$$

let $n \to \infty$

$$\int_0^{S(r,r,w)} \zeta d_p < \max\{\int_0^{S(w,w,w)} \zeta d_p$$
$$, \int_0^{\frac{1}{2}[S(r,r,w)+S(w,w,w)]} \zeta d_p$$
$$, \int_0^{\frac{1}{2}[S(w,w,w)+S(r,r,w)]} \zeta d_p\}$$

$$\int_0^{S(r,r,w)} \zeta d_p < \max\{0, \int_0^{\frac{1}{2}[S(r,r,w)+0]} \zeta d_p, \int_0^{\frac{1}{2}[0+S(r,r,w)]} \zeta d_p\}$$

$$\int_0^{S(r,r,w)} \zeta d_p < \int_0^{S(r,r,w)} \zeta d_p$$

which is contradiction since, $r = w$. Hence we have $\lim_{n\to\infty} Ay_n = w$

Suppose $P(X)$ is closed subset of X, then there exist $q \in X$ such that $Pq = w$ therefore, we have

$$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Hx_n = \lim_{n\to\infty} Px_n = w = Pq, q \in X$$

Now, from (3) we have

$$\int_0^{S(Aq,Aq,Bx_n)} \zeta d_p < \max\{\int_0^{S(Pq,Pq,Hx_n)} \zeta d_p$$
$$,\int_0^{\frac{1}{2}[S(Aq,Aq,Pq)+S(Hx_n,Hx_n,Bx_n)]} \zeta d_p$$
$$,\int_0^{\frac{1}{2}[S(Pq,Pq,Bx_n)+S(Aq,Aq,Hx_n)]} \zeta d_p\}$$

$$\lim_{n\to\infty} \parallel \int_0^{S(Aq,Aq,Bx_n)} \zeta d_p \parallel < K \parallel \max\{\int_0^{S(Pq,Pq,Hx_n)} \zeta d_p$$
$$,\int_0^{\frac{1}{2}[S(Aq,Aq,Pq)+S(Hx_n,Hx_n,Bx_n)]} \zeta d_p$$
$$,\int_0^{\frac{1}{2}[S(Pq,Pq,Bx_n)+S(Aq,Aq,Hx_n)]} \zeta d_p\} \parallel$$

Thus $S(Aq,Aq,Bx_n) = 0$ which is implies that $S(Aq,Aq,Bx_n) \ll 0$. Thus $Aq = w$ therefore, we have, $Aq = Pq = w$

since, $A(X) \subseteq H(X)$, there exists a sequence $v \in X$ such that $Aq = Hl = w$

Now, we claim that $Bl = w$, if $Bl \neq w$, then $S(w,w,Bl) > 0$ from (3), we have

$$\int_0^{S(Aq,Aq,Bl)} \zeta d_p < \max\{\int_0^{S(Pq,Pq,Hl)} \zeta d_p$$
$$,\int_0^{\frac{1}{2}[S(Aq,Aq,Pq)+S(Hl,Hl,Bl)]} \zeta d_p$$
$$,\int_0^{\frac{1}{2}[S(Pq,Pq,Bl)+S(Aq,Aq,Hl)]} \zeta d_p\}$$

$$\int_0^{S(w,w,Bl)} \zeta d_p < \max\{\int_0^{S(w,w,w)} \zeta d_p,$$
$$,\int_0^{\frac{1}{2}[S(w,w,w)+S(w,w,Bl)]} \zeta d_p$$
$$,\int_0^{\frac{1}{2}[S(w,w,Bl)+S(w,w,w)]} \zeta d_p\}$$

$$\int_0^{S(q,q,Bl)} \zeta d_p < \max\{0, \int_0^{\frac{1}{2}[0+S(w,w,Bl)]} \zeta d_p, \int_0^{\frac{1}{2}[S(w,w,Bl)+0]} \zeta d_p\}$$

which is contradiction since, $Bl = w$.

therefore, we have, thus $Hl = Bl = w$

since, the pair (A,P) is weakly compatible then, $APq = PAq \Rightarrow Aw = Pw$.

We now show that $Aw = w$ if $Aw \neq w$ then $S(Aw,w,w) > 0$ from (3) we have

$$\int_0^{S(Aw,Aw,Bl)} \zeta d_p < \max\{\int_0^{S(Pw,Pw,Hl)} \zeta d_p,$$
$$\int_0^{\frac{1}{2}[S(Aw,Aw,Pw)+S(Hl,Hl,Bl)]} \zeta d_p,$$
$$\int_0^{\frac{1}{2}[S(Pw,Pw,Bl)+S(Aw,Aw,Hl)]} \zeta d_p\}$$

$$\int_0^{S(Aw,Aw,w)} \zeta d_p < \max\{\int_0^{S(Aw,Aw,w)} \zeta d_p,$$
$$\int_0^{\frac{1}{2}[S(Aw,Aw,Aw)+S(w,w,w)]} \zeta d_p,$$
$$\int_0^{\frac{1}{2}[S(w,w,w)+S(Aw,Aw,w)]} \zeta d_p\}$$

$$\int_0^{S(Aw,Aw,w)} \zeta d_p < \max\{\int_0^{S(Aw,Aw,w)} \zeta d_p,$$
$$\int_0^{\frac{1}{2}[S(Aw,Aw,Aw)+0]} \zeta d_p,$$
$$\int_0^{\frac{1}{2}[0+S(Aw,Aw,w)]} \zeta d_p\}$$

which is contraction then $Aw = Pw = w$, now we prove w is common fixed point of A and P

Since, the pair (B,H) is weakly compatible then, $BHl = HBl \Rightarrow Bw = Hw$, we prove $Bl = l$

$$\int_0^{S(Aw,Aw,Bw)} \zeta d_p < \max\{\int_0^{S(Pw,Pw,Hw)} \zeta d_p,$$
$$\int_0^{\frac{1}{2}[S(Aw,Aw,Pw)+S(Hw,Hw,Bw)]} \zeta d_p,$$
$$\int_0^{\frac{1}{2}[S(Pw,Pw,Bw)+S(Aw,Aw,Hw)]} \zeta d_p\}$$

$$\int_0^{S(w,w,Bw)} \zeta d_p < \max\{\int_0^{S(w,w,Bw)} \zeta d_p,$$
$$\int_0^{\frac{1}{2}[S(w,w,w)+S(Bw,Bw,Bw)]} \zeta d_p,$$
$$\int_0^{\frac{1}{2}[S(w,w,Bw)+S(w,w,Bw)]} \zeta d_p\}$$

$$\int_0^{S(w,w,Bw)} \zeta d_p < \max\{\int_0^{S(w,w,Bw)} \zeta d_p, 0, \int_0^{\frac{1}{2}[S(w,w,Bw)+S(w,w,Bw)]} \zeta d_p\}$$

which is contradiction then $Bw = Hw = w$, Similarly, we can prove w is common fixed point of B and H.

Now, we prove the uniqueness of common fixed point of A, B, H and P, if $w \neq u, u \in X$, then $S(w, w, u) > 0$ from (3),

$$\int_0^{S(w,w,u)} \zeta d_p < \max\{\int_0^{S(Pw,Pw,Hw)} \zeta d_p,$$
$$\int_0^{\frac{1}{2}[S(Aw,Aw,Pw)+S(Hw,Hw,Bw)]} \zeta d_p,$$
$$\int_0^{\frac{1}{2}[S(Pw,Pw,Bw)+S(Aw,Aw,Hw)]} \zeta d_p\}$$

$$\int_0^{S(w,w,u)} \zeta d_p < \max\{\int_0^{S(w,w,u)} \zeta d_p,$$
$$\int_0^{\frac{1}{2}[S(w,w,w)+S(u,u,u)]} \zeta d_p,$$
$$\int_0^{\frac{1}{2}[S(w,w,u)+S(w,w,u)]} \zeta d_p\}$$

$$\int_0^{S(w,w,u)} \zeta d_p < \max\{\int_0^{S(w,w,u)} \zeta d_p, 0,0\}$$

which is contradiction then, then w is common fixed point.

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