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RESEARCH TITLE

On Humbert Matrix Polynomials of Three Variables

Fadhl S. N. Alsarahi[1]

[1] Department of Mathematics, Saber Faculty of Science and Education, Lahej University, Yemen.
Email: fadhlsna@gmail.com

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Abstract

This paper introduces and investigates a new class of special functions called the **Humbert matrix polynomials of three variables**. The study presents their definition via generating functions and explores their algebraic properties. A detailed examination is provided on their hypergeometric matrix representations, in addition to deriving new generating matrix functions. Furthermore, the paper establishes expansions of these polynomials in terms of well-known classical polynomials such as Legendre, Gegenbauer, Hermite, and Laguerre polynomials. These results extend existing theories of matrix orthogonal polynomials and provide a unified framework to study various families of polynomials through a matrix approach.

Key Words: Humbert matrix polynomials, three variables, hypergeometric matrix function, generating functions, orthogonal polynomial expansions.

عنوان البحث

حول كثيرات حدود همبرت المصفوفية ذات الثلاثة متغيرات

المستخلص

تتناول هذه الورقة البحثية فئة جديدة من الدوال الخاصة تُعرف باسم **كثيرات حدود همبرت المصفوفية ذات الثلاثة متغيرات** .حيث يتم تقديم تعريفها من خلال دالة التوليد، كما يتم استكشاف خصائصها الجبرية بالتفصيل. وتتناول الدراسة كذلك التمثيلات المصفوفية لها باستخدام الدوال الفائقة الهندسية (Hypergeometric) ، بالإضافة إلى اشتقاق دوال توليد إضافية. كما تقدم الورقة توسعات لهذه الكثيرات الحدودية باستخدام كثيرات حدود كلاسيكية معروفة مثل **كثيرات حدود ليجيندر، وجيغينباور، وهيرميت، ولاجور**. وتمثل هذه النتائج امتداداً للنظريات القائمة حول كثيرات الحدود المتعامدة المصفوفية، وتوفر إطاراً موحداً لدراسة عائلات متعددة من كثيرات الحدود عبر منهج مصفوفي.

الكلمات المفتاحية: كثيرات حدود همبرت المصفوفية، ثلاثة متغيرات، الدالة الفائقة الهندسية المصفوفية، دوال التوليد، توسعات كثيرات الحدود المتعامدة.

1. Introduction and Preliminaries.

The Humbert function is probably the best known special function, within pure and applied mathematics. These polynomials generalize the well known class of Gegenbauer, Legendre, Pincherl, Horadam, Horadam-Pethe and Kinney polynomials.

Gould [4] (also see [2]) presented a systematic study of an interesting generalization of Humbert, Gegenbauer and several other polynomials defined by

$$(c - mxt + yt^m)^{-p} = \sum_{n=0}^{\infty} P_n(m,x,y,p,c)t^n, \qquad (1.1)$$

where m is a positive integer and other parameters are unrestricted in general. The table of the main special cases of (1.1), includes Gegenbauer, Legender, Tcheby-cheff, Pincherle, Kinney and Humbert polynomials, (see Gould [4]).

In [9] Milovanovic and Dordevic considered the polynomials $\{P_{n,m}^{\lambda}\}_n^{\infty} = 0$ defined by the generating function

$$(1 - 2xt + t^m)^{-\lambda} = \sum_{n=0}^{\infty} P_{n,m}^{\lambda}(x)t^n, \qquad (1.2)$$

where $m \in N$ and $\lambda > -{}^1\!/_2$. Note that

$$P_{n,1}^{\lambda}(x) = \frac{(\lambda)_n}{n!}(2x-1)^n, \qquad \text{(Horadam polynomials [5])}$$

$$P_{n,2}^{\lambda}(x) = C_n^{\lambda}(x), \qquad \text{(Gegenbauer polynomials)}$$

$$P_{n,3}^{\lambda}(x) = P_{n+1}^{\lambda}(x). \qquad \text{(Horadam- pethe polynomials [6])}$$

The explicit form of the polynomial $P_{n,m}^{\lambda}(x)$ is

$$P_{n,m}^{\lambda}(x) = \sum_{k=0}^{[n/m]}(-1)^k \frac{(\lambda)_{n-(m-1)k}(2x)^{n-mk}}{k!(n-mk)!}. \qquad (1.3)$$

The set of polynomials denoted by $S_n^{v}(x)$ considered by Sinha [16]

$$\left(1 - 2xt + t^m(2x-1)\right)^{-v} = \sum_{n=0}^{\infty} S_n^{v}(x)(x)t^n, \qquad (1.4)$$

is precisely a generalization of $S_n(x)$ defined and studied by Shrestha [15].

A generalization of various polynomials mentioned above is provided by the definition

$$(c - axt + bt^m(2x-1)^d)^{-v} = \sum_{n=0}^{\infty} P_{n,m,a,b,c,d}^{v}(x)t^n = \sum_{n=0}^{\infty} \theta(x)t^n. \qquad (1.5)$$

Pathan and Khan introduced and studied of Humbert polynomials $h_{n,m}^{v}(x)$ defined by (see [11, p.56 (2.6) and (2.7))

$$h_{n,m}^{v}(x) = \sum_{k=0}^{[n/m]}(-1)^k \frac{(v)_{n-(m-1)k}(mx)^{n-mk}}{k!(n-mk)!}, \qquad (1.6)$$

$$h_{n,m}^{v}(x) = \sum_{k=0}^{\left[(n-(m-2)s)/m\right]} \sum_{s=0}^{k}(-k)_s \frac{(v)_k(2v+2k)_{n-2k-(m-2)s}\left({}^{mx}\!/_2\right)^{n-ms}}{(n-2k-(m-2)s)!k!s!}. \qquad (1.7)$$

In [13] Sayyed, Metwally and Batahan presented a study of Gegenbauer matrix polynomials defined by

$$(1 - 2xt + t^2)^{-A} = \sum_{n=0}^{\infty} C_n^{A}(x)t^n, \qquad (1.8)$$

where A is a positive stable matrix in the complex space $C^{N\times N}$ of all square matrices of common order N.

The explicit representation of the Gegenbauer matrix polynomials $C_n^{A}(x)$ has been given in [14, p.104(15)] in the form

$$. C_n^{A}(x) = \sum_{k=0}^{[n/2]}(-1)^k \frac{(A)_{n-k}}{k!(n-2k)!}(2x)^{n-mk}. \qquad (1.9)$$

Due to Rainville [12, p.181 (Theorem 65), p.283(36), p.194(4) and p.207(2)] we will exploit the following relations:

$$x^n = \frac{n!}{2^n}\sum_{k=0}^{[n/2]} \frac{(2n-4k+1)P_{n-2k}(x)}{k!\left({}^3\!/_2\right)_{n-k}}, \qquad (1.10)$$

$$\frac{(2x)^n}{n!} = \sum_{k=0}^{[n/2]} \frac{(v+n-2k)C_{n-2k}^{v}(x)}{k!(v)_{n+1-k}}, \qquad (1.11)$$

$$x^n = \sum_{k=0}^{[n/2]} \frac{n!H_{n-2k}(x)}{2^n k!(n-2k)!}, \qquad (1.12)$$

and

$$x^n = \sum_{k=0}^{n}(-1)^k \frac{n!(1+\alpha)_n}{(n-k)!(1+\alpha)_k} L_k^{(\alpha)}(x). \qquad (1.13)$$

Khammash and Shehata [8] presented a study of the Humbert matrix polynomials of two variables defined by

$$P_{n,k,m}(x,y;A)=\sum_{r=0}^{[n/m]}\sum_{j=0}^{[k/m]}(-1)^{i+j}\,(A)_{n+k+(1-m)(i+j)}\,\frac{(mx)^{n-mr}(my)^{k-mj}}{r!j!(n-mr)!(k-mj)!}.\quad(1.14)$$

Pathan , Bin-Saad and Alsarahi [10] introduced and studied of Humbert polynomials $P_{n,m}^{A}(x,y;a,b,c)$ defined by

$$P_{n,m}^{A}(x,y;a,b,c)=\sum_{k=0}^{[n/m]}(-1)^{k}\,\frac{c^{-A-(n-(m-1)k)I}(A)_{n+(m-1)k}}{k!(n-2k)!}\,(ax)^{n-mk}[b(2y-1)]^{k}.\quad(1.15)$$

In the last decade the study of matrix polynomials has been made more systematic with the consequence that many basic results of scalar orthogonality have been extended to the matrix case (see, for example [1] and [2]).

If $c\in C^{r\times r}$ is such that $C+nI$ is invertible for every integer $n\geq 0$, then

$$(c)_n=\Gamma(c+nI)\Gamma^{-1}(z).\quad(1.16)$$

If $A,B,C\in C^{r\times r}$ for which $C+nI$ is invertible for every integer $n\geq 0$. The hypergeometric matrix function $F(A,B,C;z)$ is defined by

$$F(A,B;C;z)=\sum_{n=0}^{\infty}\frac{(A)_n(B)_n[(C)_n]^{-1}}{n!}z^n,\quad(1.17)$$

it converges for $|z|>1$.

The generalized hypergeometric matrix function is given in the form:

$${}_pF_q\left(A_1,A_2,\dots,A_p;C_1,C_2,\dots,C_q;z\right)=\sum_{n=0}^{\infty}\frac{(A_1)_n(A_2)_n\dots(A_p)_n[(C_1)_n]^{-1}[(C_2)_n]^{-1}\dots\left[(C_q)_n\right]^{-1}}{n!}z^n.\quad(1.18)$$

For $A\in C^{r\times r}$, the matrix version of the pochhammer symbol (the shifted factorial) is

$$(A)_n=A(A+I)(A+2I)\dots(A+(n-1)I);\ \ n\geq 1;(A)_0=I.\quad(1.19)$$

Also, from (1.19) it is easy to see that

$$(A)_{n+k}=(A)_n(A+nI)_k,\quad(1.20)$$

$$(A)_{n-k}=(-1)^k(A)_n(I-A-nI)^{k-1},\quad(1.21)$$

and

$$(A)_{mn}=m^m\prod_{s=1}^{m}\left(\frac{1}{m}(A+(s-1)I)\right)_n,\quad(1.22)$$

where m is positive integer.

$$\frac{1}{(n-mk)!}=\frac{(-1)^{mk}}{n!}(-n)_{mk};\ 0\leq mk\leq n,\quad(1.23)$$

$$(-nI)_{mk}=m^{mk}\prod_{s=1}^{m}\left(\frac{1}{m}(s-n-1)I\right)_k.\quad(1.24)$$

For any $A\in C^{N\times N}$ we will exploit the following relation due to [13]

$$(1-x)^{-A}=\sum_{n=0}^{\infty}(A)_n\frac{x^n}{n!},\quad |x|<1,\quad(1.25)$$

in general

$$(1-x_1-x_2-\cdots-x_r)^{-A}=\sum_{n_1+n_2+\cdots+n_r=0}^{\infty}(A)_{n_1+n_2+\cdots+n_r}\frac{x_1^{n_1}x_2^{n_2}\dots x_r^{n_r}}{n_1!n_2!\dots n_r!},\quad(1.26)$$

where $|x_1+x_2+\cdots+x_r|<1$.

Also we recall that if $A(n,k)$ are matrix in $A\in C^{N\times N}$ for $n\geq 0$ and $k\geq 0$ that it follows that:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k),\quad(1.27)$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n+k),\quad(1.28)$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}A(k,n-2k),\quad(1.29)$$

and, for m is a positive integer such that $n>m$, then

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n+mk),\quad(1.30)$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]}A(k,n-mk),\quad(1.31)$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]}A(k,n-mk+k),\quad(1.32)$$

Finally, we recall the relation

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} x^{n-k} y^k. \tag{1.33}$$

The primary goal of this paper is to introduce and study a new class of matrix polynomials, namely the Humbert Matrix polynomials of three variables and discuss its special cases, hypergeometric matrix representations, the Additional generating matrix functions and expansions of the Humbert matrix polynomials of three variables in series of Legender, Gegenbuaer, Hermite and Leguerre polynomials are given.

2. The Humbert Matrix Polynomials Of Three Variables

Let A be a positive stable matrix in $C^{N\times N}$ and a,b,c,m are positive integers. The Humbert matrix polynomials of three variables is define by means of the generating relation:

$$\left(1-(axt-t^m)-(bys-s^m)-(czu-u^m)\right)^{-A}$$
$$= \sum_{n,k,r=0}^{\infty} P_{n,m,k,r}^{A}(a,b,c;x,y,z) t^n s^k u^r. \tag{2.1}$$

Now, by using the above generating function we will deduce the Humbert matrix polynomials of three variables $P_{n,m,k,r}^{A}(a,b,c;x,y,z)$ in the form of the following theorem:

Theorem 2.1.

Let us assume that $A \in C^{N\times N}$ and a,b,c,m are positive integers, then the following formula for the Humbert polynomials of three variables $P_{n,m,k,r}^{A}(a,b,c;x,y,z)$ holds true:

$$P_{n,m,k,r}^{A}(a,b,c;x,y,z) = \sum_{i=0}^{[n/m]} \sum_{j=0}^{[k/m]} \sum_{v=0}^{[r/m]} (-1)^{i+j+v} (A)_{(n++k+r)-(m-1)(i+j+v)}$$
$$\times \frac{(ax)^{n-mi}(by)^{k-mj}(cz)^{r-mv}}{(n-mi)!i!(k-mj)!j!(r-mv)!v!}. \tag{2.2}$$

Proof. Let us denote the left hand side of (2.1) by W and by using (1.26), we have

$$W = \left(1-(axt-t^m)-(bys-s^m)-(czu-u^m)\right)^{-A}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty} (A)_{n+k+r} \frac{(axt-t^m)^n (bys-s^m)^k (czu-u^m)^r}{n!\,k!\,r!}$$

Appling relation (1.33), we obtain

$$W = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{i=0}^{n}\sum_{j=0}^{k}\sum_{v=0}^{r} (-1)^{i+j+v} (A)_{n+k+r} \frac{(axt)^{n-i}(t^m)^i}{(n-i)!\,i!}$$
$$\times \frac{(bys)^{k-j}(s^m)^j}{(k-j)!\,j!} \frac{(czu)^{r-v}(u^m)^v}{(r-v)!\,v!}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{i=0}^{n}\sum_{j=0}^{k}\sum_{v=0}^{r} (-1)^{i+j+v} (A)_{n+k+r} \frac{(ax)^{n-i}(by)^{k-j}}{(n-i)!\,i!}$$
$$\times \frac{(cz)^{r-v}}{(k-j)!\,j!\,(r-v)!\,v!} t^{n+(m-1)i} s^{k+(m-1)j} u^{r+(m-1)v}$$

which using relation (1.28), we find

$$W = \sum_{n,k,r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{v=0}^{\infty} (-1)^{i+j+v} (A)_{n+i+k+j+r+v} \frac{(ax)^n (by)^k (cz)^r}{n!\,i!\,k!\,j!\,r!\,v!}$$
$$\times t^{n+mi} s^{k+mj} u^{r+mv}$$

Using relation (1.31), we obtain

$$W = \sum_{n,k,r=0}^{\infty}\sum_{i=0}^{[n/m]}\sum_{j=0}^{[k/m]}\sum_{v=0}^{[r/m]} (-1)^{i+j+v} (A)_{n+(1-m)i+k+(1-m)j+r+(1-m)v}$$

$$\times \frac{(ax)^{n-mi}(by)^{k-mj}(cz)^{r-mv}}{(n-mi)!i!(k-mj)!j!(r-mv)!v!} t^n s^k u^r. \quad (2.3)$$

By equating the coefficients of $t^n s^k u^r$ with the right hand side of (2.1), we get the relation (2.2).

Putting $z = 0, a = b = m$ in equation (2.2) and in view of equation (1.14).

3. Hypergeometric matrix representations

We study here the representation of the hypergeometric matrix representation for the Humbert matrix polynomials of three variables by using relation (1.23) in (2.2), we get

$$P^A_{n,m,k,r}(a,b,c;x,y,z) = \sum_{i=0}^{[n/m]} \sum_{j=0}^{[k/m]} \sum_{v=0}^{[r/m]} (-1)^{(m+1)(i+j+v)} (A)_{(n++k+r)-(m-1)(i+j+v)} \times \frac{(-n)_{mi}(-k)_{mj}(-r)_{mv}(ax)^{n-mi}(by)^{k-mj}(cz)^{r-mv}}{n!i!k!j!r!v!}, \quad (3.1)$$

Also, using relations (1.24) and (1.22), we obtain

$$P^A_{n,m,k,r}(a,b,c;x,y,z) = \frac{(A)_{(n++k+r)}(ax)^n(by)^k(cz)^r}{n!\,k!\,r!} \sum_{i=0}^{[n/m]} \sum_{j=0}^{[k/m]} \sum_{v=0}^{[r/m]} \frac{1}{i!\,j!\,v!} \prod_{s=1}^{m} \left(\frac{(s-n-1)}{m} I\right)_i \times \prod_{l=1}^{m} \left(\frac{(l-k-1)}{m} I\right)_j \prod_{u=1}^{m} \left(\frac{(u-r-1)}{m} I\right)_v \left[\prod_{s=1}^{m-1} \left(\frac{-A-(n+k+r)I+sI}{m-1}\right)_{i+j+v}\right]^{-1} \times \left(\frac{m^m}{(m-1)^{m-1}(ax)^m}\right)^i \left(\frac{m^m}{(m-1)^{m-1}(by)^m}\right)^j \left(\frac{m^m}{(m-1)^{m-1}(cz)^m}\right)^v, \quad (3.2)$$

which on using the definition of the generalized hypergeometric series ${}_pF_q$ [17] gives us the following hypergeometric matrix representation

$$P^A_{n,m,k,r}(a,b,c;x,y,z) = \frac{(A)_{(n++k+r)}(ax)^n(by)^k(cz)^r}{n!\,k!\,r!} {}_mF^{(3)}_{m-1}\Big[\frac{n}{m}I, \frac{n-1}{m}I, \dots, \times \frac{n-m-1}{m}I, \frac{k}{m}I, \frac{k-1}{m}I, \dots, \frac{k-m-1}{m}I, \frac{r}{m}I, \frac{r-1}{m}I, \dots, \frac{r-m-1}{m}I; \frac{-A-((n+k+r)-1)I}{m-1}, \frac{-A-((n+k+r)-2)I}{m-1}, \dots, \frac{-A-((n+k+r)-(m-1))I}{m-1} ; \frac{m^m}{(m-1)^{m-1}(ax)^m}, \frac{m^m}{(m-1)^{m-1}(by)^m}, \frac{m^m}{(m-1)^{m-1}(cz)^m}\Big]. \quad (3.3)$$

For $z = 0, a = b = m$, we gives hypergeometric representation of Humbert matrix polynomials of two variables [8].

4. Additional Generating Matrix Functions

By proceeding in a fashion similar to that in Section 2, in this section we aim at establishing the following additional generating functions for the Humbert matrix polynomials $P^A_{n,m,k,r}(a,b,c;x,y,z)$:

$$\sum_{n,k,r=0}^{\infty} P^A_{n,m,k,r}(a,b,c;x,y,z)[(A)_{n+k+r}]^{-1} t^n s^k u^r = \sum_{n,k,r=0}^{\infty} \frac{(axt)^n(bys)^k(czu)^r}{n!\,k!\,r!} {}_1F^{(3)}_m\Big[A+(n+k+r)I; \frac{A+(n+k+r)I}{m}, \dots, \frac{A+(n+k+r-(m-1))I}{m}; -\left(\frac{t}{m}\right)^m, -\left(\frac{s}{m}\right)^m, -\left(\frac{u}{m}\right)^m\Big], \quad (4.1)$$

$$\sum_{n,k,r=0}^{\infty} (e)_{n+k+r} P^A_{n,m,k,r}(a,b,c;x,y,z)[(A)_{n+k+r}]^{-1} t^n s^k u^r$$

$$= \sum_{n,k,r=0}^{\infty} \frac{(e)_{n+k+r}(axt)^n(bys)^k(czu)^r}{n!\,k!\,r!} {}_{m+1}F_m^{(3)}[A+(n+k+r)I;$$
$$\frac{e+(n+k+r)}{m}, \dots, \frac{e+\left(n+k+r-(m-1)\right)}{m}; \frac{A+(n+k+r)I}{m}, \dots$$
$$\dots, \frac{A+\left(n+k+r-(m-1)\right)I}{m}; -\left(\frac{t}{m}\right)^m, -\left(\frac{s}{m}\right)^m, -\left(\frac{u}{m}\right)^m], \qquad (4.2)$$

Proof. From (2.2), we have

$$\sum_{n,k,r=0}^{\infty} P_{n,m,k,r}^{A}(a,b,c;x,y,z)[(A)_{n+k+r}]^{-1} t^n s^k u^r$$
$$= \sum_{n,k,r=0}^{\infty} \sum_{i=0}^{[n/m]} \sum_{j=0}^{[k/m]} \sum_{v=0}^{[r/m]} (-1)^{i+j+v} (A)_{(n+k+r)+(1-m)(i+j+v)} [(A)_{n+k+r}]^{-1}$$
$$\times \frac{(ax)^{n-mi}(by)^{k-mj}(cz)^{r-mv}}{(n-mi)!\,i!\,(k-mj)!\,j!\,(r-mv)!\,v!} t^n s^k u^r$$

Now, on using the result (1.29), we get

$$\sum_{n,k,r=0}^{\infty} P_{n,m,k,r}^{A}(a,b,c;x,y,z)[(A)_{n+k+r}]^{-1} t^n s^k u^r$$
$$= \sum_{n,k,r=0}^{\infty} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{v=0}^{\infty} (-1)^{i+j+v} (A)_{(n+k+r)+(i+j+v)} \left[(A)_{(n+k+r)+m(i+j+v)}\right]^{-1}$$
$$\times \frac{(ax)^n(by)^k(cz)^r}{n!\,i!\,k!\,j!\,r!\,v!} t^{n+mi} s^{k+mj} u^{r+mv}$$
$$= \sum_{n,k,r=0}^{\infty} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{v=0}^{\infty} (-1)^{i+j+v} \left[(A)_{n+k+r}(A+(n+k+r)I)_{m(i+j+v)}\right]^{-1}$$
$$\times (A)_{n+k+r}(A+(n+k+r)I)_{i+j+v} \frac{(ax)^n(by)^k(cz)^r}{n!\,i!\,k!\,j!\,r!\,v!} t^{n+mi} s^{k+mj} u^{r+mv}$$
$$= \sum_{n,k,r=0}^{\infty} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{v=0}^{\infty} (-1)^{i+j+v} (A+(n+k+r)I)_{i+j+v}$$
$$\times \left[(A+(n+k+r)I)_{m(i+j+v)}\right]^{-1} \frac{(ax)^n(by)^k(cz)^r}{n!\,i!\,k!\,j!\,r!\,v!} t^{n+mi} s^{k+mj} u^{r+mv}$$

And by using (1.21), we get

$$\sum_{n,k,r=0}^{\infty} P_{n,m,k,r}^{A}(a,b,c;x,y,z)[(A)_{n+k+r}]^{-1} t^n s^k u^r$$
$$= \sum_{n,k,r=0}^{\infty} \frac{(axt)^n(bys)^k(czu)^r}{n!\,k!\,r!} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{v=0}^{\infty} \frac{(-1)^{i+j+v}}{i!\,j!\,v!} (A+(n+k+r)I)_{i+j+v}$$
$$\times \left[m^{m(i+j+v)} \prod_{w=0}^{m} \left(\frac{A+(n+k+r-w)I}{m}\right)_{i+j+v}\right]^{-1} (t^m)^i (s^m)^j (u^m)^v$$
$$= \sum_{n,k,r=0}^{\infty} \frac{(axt)^n(bys)^k(czu)^r}{n!\,k!\,r!} \sum_{i,j,v=0}^{\infty} \frac{(-1)^{i+j+v}}{i!\,j!\,v!} (A+(n+k+r)I)_{i+j+v}$$
$$\times \left[\prod_{w=0}^{m} \left(\frac{A+(n+k+r-w)I}{m}\right)_{i+j+v}\right]^{-1} \left(\left(\frac{t}{m}\right)^m\right)^i \left(\left(\frac{s}{m}\right)^m\right)^j \left(\left(\frac{u}{m}\right)^m\right)^v$$

which on using the definition of the generalized hypergeometric series ${}_pF_q$ [45] , gives us the hypergeometric matrix representation in (4.1).

If e is an arbitrary number, maybe a complex number, in similar way (4.1), we can obtain (4.2).

5. Expansions

From (2.2), we have

$$\sum_{n,k,r=0}^{\infty} P_{n,m,k,r}^{A}(a,b,c;x,y,z)t^n s^k u^r = \sum_{n,k,r=0}^{\infty}\sum_{i=0}^{[n/m]}\sum_{j=0}^{[k/m]}\sum_{v=0}^{[r/m]}(-1)^{i+j+v}$$
$$\times (A)_{(n+k+r)+(1-m)(i+j+v)}\frac{(ax)^{n-mi}(by)^{k-mj}(cz)^{r-mv}}{(n-mi)!i!(k-mj)!j!(r-mv)!v!}t^n s^k u^r, \qquad (5.1)$$

Using relation (1.29), we find

$$\sum_{n,k,r=0}^{\infty} P_{n,m,k,r}^{A}(a,b,c;x,y,z)t^n s^k u^r = \sum_{n,k,r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{v=0}^{\infty}(-1)^{i+j+v}$$
$$\times (A)_{(n+k+r)+(i+j+v)}\frac{(ax)^n(by)^k(cz)^r}{n!i!k!j!r!v!}t^{n+mi}s^{k+mj}u^{r+mv}, \qquad (5.2)$$

On using the result (1.10), we have

$$\frac{(ax)^n}{n!} = \sum_{s=0}^{[n/2]}\frac{(2n-4s+1)}{s!\,(3/2)_{n-s}}P_{n-2s}(ax/2)$$

equation (5.2) gives us

$$\sum_{n,k,r=0}^{\infty} P_{n,m,k,r}^{A}(a,b,c;x,y,z)t^n s^k u^r = \sum_{n,k,r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{v=0}^{\infty}\sum_{s=0}^{[n/2]}\sum_{l=0}^{[k/2]}\sum_{w=0}^{[r/2]}(-1)^{i+j+v}$$
$$\times (A)_{(n+k+r)+(i+j+v)}\frac{(2n-4s+1)(2k-4u+1)(2r-4w+1)}{i!\,j!\,v!\,s!\,(3/2)_{n-s}l!\,(3/2)_{k-l}w!\,(3/2)_{r-w}}$$
$$\times P_{n-2s}(ax/2).P_{k-2l}(by/2).P_{r-2w}(cz/2).t^{n+mi}s^{k+mj}u^{r+mv}$$

Using relation (1.30), we obtain

$$\sum_{n,k,r=0}^{\infty} P_{n,m,k,r}^{A}(a,b,c;x,y,z)t^n s^k u^r = \sum_{n,k,r=0}^{\infty}\sum_{i,j,v=0}^{\infty}\sum_{s,l,w=0}^{\infty}(-1)^{i+j+v}$$
$$\times (A)_{(n+k+r)+2(s+l+w)+(i+j+v)}\frac{(2n+1)(2k+1)(2r+1)}{i!\,j!\,v!\,s!\,(3/2)_{n+s}l!\,(3/2)_{k+l}w!\,(3/2)_{r+w}}$$
$$\times P_n(ax/2).P_k(by/2).P_r(cz/2).t^{n+2s+mi}s^{k+2l+mj}u^{r+2w+mv}$$

By using relation (1.27), we get

$$\sum_{n,k,r=0}^{\infty} P_{n,m,k,r}^{A}(a,b,c;x,y,z)t^n s^k u^r = \sum_{n,k,r=0}^{\infty}\sum_{i,j,v=0}^{\infty}\sum_{s=0}^{i}\sum_{l=0}^{j}\sum_{w=0}^{v}(-1)^{i+j+v-(s+l+w)}$$
$$\times (A)_{(n+k+r)+(s+l+w)+(i+j+v)}\frac{(2n+1)(2k+1)(2r+1)}{(i-s)!\,(j-l)!\,(v-w)!}\frac{P_n(ax/2).P_k(by/2)}{s!\,(3/2)_{n+s}l!\,(3/2)_{k+l}}$$
$$\times\frac{P_r(cz/2)}{w!\,(3/2)_{r+w}}t^{n+mi+(2-m)s}s^{k+mj+(2-m)l}u^{r+mv+(2-m)w}$$

Using series manipulation, for which n,k,r can be replaced by $n-mi-(2-m)s, k-mj-(2-m)l, r-mv-(2-m)w$ respectively in the right-hand side of the last equation, so this equation can be written as:

$$\sum_{n,k,r=0}^{\infty} P_{n,m,k,r}^{A}(a,b,c;x,y,z)t^n s^k u^r = \sum_{n,k,r=0}^{\infty}\sum_{i=0}^{\left[\frac{n-(m-2)s}{m}\right]}\sum_{j=0}^{\left[\frac{k-(m-2)l}{m}\right]}\sum_{v=0}^{\left[\frac{r-(m-2)w}{m}\right]}$$

$$\times \sum_{s=0}^{i}\sum_{l=0}^{j}\sum_{w=0}^{v}(-1)^{i+j+v-(s+l+w)}\,(A)_{(n+k+r)+(1-m)(i+j+v)-(1-m)(s+l+w)}$$
$$(2n-2mi-2(2-m)s+1)(2k-2mj-2(2-m)l+1)(2r-2mv-2(2-m)w+1)$$
$$\times \frac{(-i)_s(-j)_l(-v)_w}{i!j!v!}\frac{P_{n-mi-(2-m)s}(ax/2).P_{k-mj-(2-m)l}(by/2)P_{r-mv-(2-m)w}(cz/2)}{s!(3/2)_{n-mi-(1-m)s}l!(3/2)_{k-mj-(1-m)l}w!(3/2)_{r-mv-(2-m)w}}t^n s^k u^r. \quad (5.3)$$

By comparing the coefficients of $t^n s^k u^r$ in both sides of (5.3), we get

$P^{A}_{n,m,k,r}(a,b,c;x,y,z)$

$$= \sum_{i=0}^{\left[\frac{n+(m-2)s}{m}\right]}\sum_{j=0}^{\left[\frac{k+(m-2)l}{m}\right]}\sum_{v=0}^{\left[\frac{r+(m-2)w}{m}\right]}\sum_{s=0}^{i}\sum_{l=0}^{j}\sum_{w=0}^{v}(-1)^{i+j+v-(s+l+w)}$$
$$\times (A)_{(n+k+r)+(1-m)(i+j+v)+(m-1)(s+l+w)}(2n-2mi-2(2-m)s+1)$$
$$\times (2k-2mj-2(2-m)l+1)(2r-2mv-2(2-m)w+1)\frac{(-i)_s(-j)_l(-v)_w}{i!\,j!\,v!}$$
$$\times \frac{P_{n-mi-(2-m)s}(ax/2).P_{k-mj-(2-m)l}(by/2)P_{r-mv-(2-m)w}(cz/2)}{s!(3/2)_{n-mi-(1-m)s}l!(3/2)_{k-mj-(1-m)l}w!(3/2)_{r-mv-(2-m)w}}. \quad (5.4)$$

The result (5.4) is an expanded for Humbert matrix polynomial in a series of Legender polynomials. Similarly, on using the result (1.11), which can be written as

$$\frac{(ax)^n}{n!} = \frac{\left(2\frac{ax}{2}\right)^n}{n!} = \sum_{s=0}^{[n/2]}\frac{(A+(n-2s)I)}{s!}[(A)_{n+1-s}]^{-1}C^{A}_{n-2s}(ax/2). \quad (5.5)$$

Now, by using (5.5) in (5.2), we get

$$\sum_{n,k,r=0}^{\infty}P^{A}_{n,m,k,r}(a,b,c;x,y,z)t^n s^k u^r = \sum_{n,k,r=0}^{\infty}\sum_{i,j,v=0}^{\infty}\frac{(-1)^{i+j+v}}{i!\,j!\,v!}(A)_{(n+k+r)+(i+j+v)}$$
$$\times \sum_{s=0}^{[n/2]}\sum_{l=0}^{[k/2]}\sum_{w=0}^{[r/2]}\frac{(A+(n-2s)I)(A+(k-2l)I)(A+(r-2w)I)}{s!\,l!\,w!}[(A)_{n+1-s}]^{-1}$$
$$[(A)_{k+1-l}]^{-1}[(A)_{r+1-w}]^{-1}C^{A}_{n-2s}(ax/2)C^{A}_{k-2l}(by/2)\,C^{A}_{r-2w}(cz/2)t^{n+mi}s^{k+mj}u^{r+mv} \quad (5.6)$$

By using (1.30) and (1.27) in (5.6) respectively, we get

$$\sum_{n,k,r=0}^{\infty}P^{A}_{n,m,k,r}(a,b,c;x,y,z)t^n s^k u^r = \sum_{n,k,r=0}^{\infty}\sum_{i,j,v=0}^{\infty}\sum_{s=0}^{i}\sum_{l=0}^{j}\sum_{w=0}^{v}$$
$$\times \frac{(-1)^{i+j+v-(s+l+w)}(A)_{(n+k+r)+(i+j+v)+(s+l+w)}}{(i-s)!\,(j-l)!\,(v-w)!\,s!\,l!\,w!}(A+nI)(A+kI)(A+rI)$$
$$\times [(A)_{n+1+s}]^{-1}[(A)_{k+1+l}]^{-1}[(A)_{r+1+w}]^{-1}C^{A}_{n}(ax/2)C^{A}_{k}(by/2)\,C^{A}_{r}(cz/2)$$
$$\times t^{n+2s+m(i-s)}s^{k+2l+m(j-l)}u^{r+2w+m(v-w)}. \quad (5.7)$$

Using series manipulation, for which n,k,r can be replaced by $n-mi-(2-m)s, k-mj-(2-m)l, r-mv-(2-m)w$ respectively in the right-hand side of the last equation, so this equation can be written as:

$$\sum_{n,k,r=0}^{\infty}P^{A}_{n,m,k,r}(a,b,c;x,y,z)t^n s^k u^r = \sum_{n,k,r=0}^{\infty}\sum_{i=0}^{\left[\frac{n+(m-2)s}{m}\right]}\sum_{j=0}^{\left[\frac{k+(m-2)l}{m}\right]}\sum_{v=0}^{\left[\frac{r+(m-2)w}{m}\right]}$$
$$\times \sum_{s=0}^{i}\sum_{l=0}^{j}\sum_{w=0}^{v}(-1)^{i+j+v-(s+l+w)}\frac{(A)_{(n+k+r)-(m-1)(i+j+v)+(m-1)(s+l+w)}}{(i-s)!\,(j-l)!\,(v-w)!\,s!\,l!\,w!}$$
$$\times (A+(n-mi+(m-2)s)I)\left[(A)_{n-mi+(m-1)s+1}\right]^{-1}C^{A}_{n-mi+(m-2)s}(ax/2)$$
$$\times (A+(k-mj+(m-2)l)I)\left[(A)_{k-mj+(m-1)l+1}\right]^{-1}C^{A}_{k-mj+(m-2)l}(by/2)$$
$$\times (A+(r-mv+(m-2)w)I)\left[(A)_{r-mv+(m-1)w+1}\right]^{-1}C^{A}_{r-mv+(m-2)w}(cz/2)$$

$$\times t^n s^k u^r. \tag{5.7}$$

By equating the coefficients of $t^n s^k u^r$, we obtain

$$P^A_{n,m,k,r}(a,b,c;x,y,z) = \sum_{i=0}^{\left[\frac{n+(m-2)s}{m}\right]} \sum_{j=0}^{\left[\frac{k+(m-2)l}{m}\right]} \sum_{v=0}^{\left[\frac{r+(m-2)w}{m}\right]} \sum_{s=0}^{i}\sum_{l=0}^{j}\sum_{w=0}^{v} (-1)^{i+j+v-(s+l+w)}$$
$$\times \frac{(A)_{(n+k+r)-(m-1)(i+j+v)+(m-1)(s+l+w)}}{(i-s)!\,(j-l)!\,(v-w)!\,s!\,l!\,w!}(A+(n-mi+(m-2)s)I)$$
$$\times \left[(A)_{n-mi+(m-1)s+1}\right]^{-1} C^A_{n-mi+(m-2)s}\left({}^{ax}/_2\right)(A+(k-mj+(m-2)l)I)$$
$$\times \left[(A)_{k-mj+(m-1)l+1}\right]^{-1} C^A_{k-mj+(m-2)l}\left({}^{by}/_2\right)(A+(r-mv+(m-2)w)I)$$
$$\times \left[(A)_{r-mv+(m-1)w+1}\right]^{-1} C^A_{r-mv+(m-2)w}\left({}^{cz}/_2\right). \tag{5.8}$$

The result (5.8) is an expanded for Humbert matrix polynomial in a series of Gegenbuaer polynomials.

On using the result (1.12), which can be written as

$$(ax)^n = \left(2\frac{ax}{2}\right)^n = \sum_{s=0}^{[n/2]} \frac{n!}{s!(n-2s)!} H_{n-2s}\left({}^{ax}/_2\right), \tag{5.9}$$

Now, put (5.9) in (5.2), we find

$$\sum_{n,k,r=0}^{\infty} P^A_{n,m,k,r}(a,b,c;x,y,z)t^n s^k u^r = \sum_{n,k,r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{v=0}^{\infty}(-1)^{i+j+v}$$
$$\times \sum_{s=0}^{[n/2]}\sum_{l=0}^{[k/2]}\sum_{w=0}^{[r/2]} \frac{(A)_{(n+k+r)+(i+j+v)}}{i!\,j!\,v!\,s!\,(n-2s)!\,l!\,(k-2l)!\,w!\,(r-2w)!} H_{n-2s}\left({}^{ax}/_2\right)$$
$$\times H_{k-2l}\left({}^{by}/_2\right) H_{r-2w}\left({}^{cz}/_2\right) t^{n+mi} s^{k+mj} u^{r+mv}, \tag{5.10}$$

By using (1.30), (1.27) and series manipulation, for which n,k,r can be replaced by $n-mi-(2-m)s, k-mj-(2-m)l, r-mv-(2-m)w$ respectively in the right-hand side of the last equation, so this equation can be written as:

$$\sum_{n,k,r=0}^{\infty} P^A_{n,m,k,r}(a,b,c;x,y,z)t^n s^k u^r = \sum_{n,k,r=0}^{\infty} \sum_{i=0}^{\left[\frac{n+(m-2)s}{m}\right]} \sum_{j=0}^{\left[\frac{k+(m-2)l}{m}\right]} \sum_{v=0}^{\left[\frac{r+(m-2)w}{m}\right]}$$
$$\times \sum_{s=0}^{i}\sum_{l=0}^{j}\sum_{w=0}^{v} (-1)^{i+j+v-(s+l+w)} \frac{(A)_{(n+k+r)-(m-1)(i+j+v)+(m-1)(s+l+w)}}{s!\,(i-s)!\,l!\,(j-l)!\,w!\,(v-w)!}$$
$$\times \frac{H_{n-mi+(m-2)s}\left({}^{ax}/_2\right) H_{k-mj+(m-2)l}\left({}^{by}/_2\right) H_{r-mv+(m-2)w}\left({}^{cz}/_2\right)}{(n-mi+(m-2)s)!(k-mj+(m-2)l)!(r-mv+(m-2)w)!} t^n s^k u^r, \tag{5.11}$$

By equation the coefficients of $t^n s^k u^r$ in (5.11), we get

$$P^A_{n,m,k,r}(a,b,c;x,y,z) = \sum_{i=0}^{\left[\frac{n+(m-2)s}{m}\right]} \sum_{j=0}^{\left[\frac{k+(m-2)l}{m}\right]} \sum_{v=0}^{\left[\frac{r+(m-2)w}{m}\right]}$$
$$\times \sum_{s=0}^{i}\sum_{l=0}^{j}\sum_{w=0}^{v} (-1)^{i+j+v-(s+l+w)} \frac{(A)_{(n+k+r)-(m-1)(i+j+v)+(m-1)(s+l+w)}}{s!\,(i-s)!\,l!\,(j-l)!\,w!\,(v-w)!}$$
$$\times \frac{H_{n-mi+(m-2)s}\left({}^{ax}/_2\right) H_{k-mj+(m-2)l}\left({}^{by}/_2\right) H_{r-mv+(m-2)w}\left({}^{cz}/_2\right)}{(n-mi+(m-2)s)!(k-mj+(m-2)l)!(r-mv+(m-2)w)!}. \tag{5.12}$$

The result (5.12) is an expanded for Humbert matrix polynomial in a series of Hermite polynomials.
Also, on using the result (1.13), which can be written

$$(ax)^n = \left(2\frac{ax}{2}\right)^n = 2^n\left(\frac{ax}{2}\right)^n = 2^n \sum_{s=0}^{n}(-1)^s \frac{n!(1+\alpha)_n}{(n-s)!(1+\alpha)_s} L_s^{\alpha}\left({}^{ax}/_{2}\right), \tag{5.13}$$

Now, by using (5.13) in (5.2), we get

$$\sum_{n,k,r=0}^{\infty} P_{n,m,k,r}^{A}(a,b,c;x,y,z)t^n s^k u^r = \sum_{n,k,r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{v=0}^{\infty}(-1)^{i+j+v}\, 2^{n+k+r}$$
$$\times \sum_{s=0}^{n}\sum_{l=0}^{k}\sum_{w=0}^{r}\frac{(A)_{(n+k+r)+(i+j+v)}(1+\alpha)_n(1+\alpha)_k(1+\alpha)_r}{i!\,j!\,v!\,(n-s)!\,(k-l)!\,(r-w)!\,(1+\alpha)_s(1+\alpha)_l(1+\alpha)_w}$$
$$\times L_s^{\alpha}\left({}^{ax}/_{2}\right)L_l^{\alpha}\left({}^{ax}/_{2}\right)L_w^{\alpha}\left({}^{ax}/_{2}\right)t^{n+mi}s^{k+mj}u^{r+mv}, \tag{5.14}$$

By using (1.32) and series manipulation, for which n,k,r can be replaced by $n-s-mi, k-l-mj, r-w-mv$ in the right-hand side of the last equation, so this equation can be written as:

$$\sum_{n,k,r=0}^{\infty} P_{n,m,k,r}^{A}(a,b,c;x,y,z)t^n s^k u^r = \sum_{n,k,r=0}^{\infty}\sum_{i=0}^{\left[\frac{n-s}{m}\right]}\sum_{j=0}^{\left[\frac{k-l}{m}\right]}\sum_{v=0}^{\left[\frac{r-w}{m}\right]}\sum_{s=0}^{\left[{}^{n}/_{2}\right]}\sum_{l=0}^{\left[{}^{k}/_{2}\right]}\sum_{w=0}^{\left[{}^{r}/_{2}\right]}$$
$$\times (-1)^{i+j+v}\frac{2^{n+k+r-m(i+j+v)}(A)_{(n+k+r)-(m-1)(i+j+v)}}{i!\,j!\,v!\,(n-s-mi)!\,(k-l-mj)!\,(r-w-mv)!}$$
$$\times \frac{(1+\alpha)_{n-mi}(1+\alpha)_{k-mj}(1+\alpha)_{r-mv}}{(1+\alpha)_s(1+\alpha)_l(1+\alpha)_w} L_s^{\alpha}\left({}^{ax}/_{2}\right)L_l^{\alpha}\left({}^{ax}/_{2}\right)L_w^{\alpha}\left({}^{ax}/_{2}\right)t^n s^k u^r, \tag{5.15}$$

By comparing the coefficients of $t^n s^k u^r$ in (5.15), we obtain

$$P_{n,m,k,r}^{A}(a,b,c;x,y,z) = \sum_{i=0}^{\left[\frac{n-s}{m}\right]}\sum_{j=0}^{\left[\frac{k-l}{m}\right]}\sum_{v=0}^{\left[\frac{r-w}{m}\right]}\sum_{s=0}^{\left[{}^{n}/_{2}\right]}\sum_{l=0}^{\left[{}^{k}/_{2}\right]}\sum_{w=0}^{\left[{}^{r}/_{2}\right]}(-1)^{i+j+v}2^{n+k+r-m(i+j+v)}$$
$$\times \frac{(A)_{(n+k+r)-(m-1)(i+j+v)}(1+\alpha)_{n-mi}(1+\alpha)_{k-mj}}{i!\,j!\,v!\,(n-s-mi)!\,(k-l-mj)!\,(r-w-mv)!}$$
$$\times \frac{(1+\alpha)_{r-mv}}{(1+\alpha)_s(1+\alpha)_l(1+\alpha)_w} L_s^{\alpha}\left({}^{ax}/_{2}\right)L_l^{\alpha}\left({}^{ax}/_{2}\right)L_w^{\alpha}\left({}^{ax}/_{2}\right). \tag{5.16}$$

The result (5.16) is an expanded for Humbert matrix polynomial in a series of Leguerre polynomials.

Finally, we will expand the generalized Humbert matrix polynomials in series of the generalized Hermite matrix polynomials by employing (2.3), (1.30) and [37,p.276 (3.3)] and taking into account that each matrix commutes with itself, one get

$$\sum_{n,k,r=0}^{\infty} a^{-n}b^{-k}c^{-r}\left(\sqrt{2A}\right)^{n+k+r}P_{n,m,k,r}^{A}(a,b,c;x,y,z)t^n s^k u^r$$
$$= \sum_{n,k,r=0}^{\infty}\sum_{i,j,v=0}^{\infty}\sum_{s=0}^{\left[{}^{n}/_{m}\right]}\sum_{l=0}^{\left[{}^{k}/_{m}\right]}\sum_{w=0}^{\left[{}^{r}/_{m}\right]}(-1)^{i+j+v}\,(A)_{(n+k+r)+(i+j+v)}$$
$$\times \frac{H_{n-ms,m}(x,A)H_{k-ml,m}(y,A)H_{r-mw,m}(z,A)}{i!\,j!\,v!\,s!\,l!\,w!}t^{n+mi}s^{k+mj}u^{r+mv}$$
$$= \sum_{n,k,r=0}^{\infty}\sum_{i,j,v=0}^{\infty}\sum_{s,l,w=0}^{\infty}(-1)^{i+j+v}\,(A)_{(n+k+r)+(i+j+v)+m(s+l+w)}$$
$$\times \frac{H_{n,m}(x,A)H_{k,m}(y,A)H_{r,m}(z,A)}{i!\,j!\,v!\,s!\,l!\,w!}t^{n+mi+ms}s^{k+mj+ml}u^{r+mv+mw}$$

Also, using relation (1.27), we find

$$\sum_{n,k,r=0}^{\infty} a^{-n}b^{-k}c^{-r}\left(\sqrt{2A}\right)^{n+k+r}P_{n,m,k,r}^{A}(a,b,c;x,y,z)t^n s^k u^r$$
$$= \sum_{n,k,r=0}^{\infty}\sum_{i,j,v=0}^{\infty}\sum_{s=0}^{i}\sum_{l=0}^{j}\sum_{w=0}^{v}(-1)^{i+j+v-(s+l+w)}\,(A)_{(n+k+r)+(i+j+v)+(m-1)(s+l+w)}$$

$$\times \frac{H_{n,m}(x,A)H_{k,m}(y,A)H_{r,m}(z,A)}{(i-s)!\,(j-l)!\,(v-w)!\,s!\,l!\,w!} t^{n+mi} s^{k+mj} u^{r+mv}$$

$$= \sum_{n,k,r=0}^{\infty} \sum_{i,j,v=0}^{\infty} \sum_{s=0}^{i} \sum_{l=0}^{j} \sum_{w=0}^{v} (-1)^{i+j+v-(s+l+w)} (A)_{(n+k+r)+(i+j+v)+(m-1)(s+l+w)}$$

$$\times \frac{H_{n,m}(x,A)H_{k,m}(y,A)H_{r,m}(z,A)(-iI)_s(-jI)_l(-vI)_w}{(-1)^{s+l+w} i!\,j!\,v!\,s!\,l!\,w!} t^{n+mi} s^{k+mj} u^{r+mv}$$

which by using series manipulation, for which n, k, r can be replaced by $n-mi, k-ml, r-mv$ respectively in the right-hand side of the last equation, so this equation can be written as:

$$\sum_{n,k,r=0}^{\infty} a^{-n} b^{-k} c^{-r} \left(\sqrt{2A}\right)^{n+k+r} P^{A}_{n,m,k,r}(a,b,c;x,y,z) t^n s^k u^r = \sum_{n,k,r=0}^{\infty} \sum_{i=0}^{[n/m]} \sum_{j=0}^{[k/m]} \sum_{v=0}^{[r/m]}$$

$$\times \sum_{s=0}^{i} \sum_{l=0}^{j} \sum_{w=0}^{v} (-1)^{i+j+v-(s+l+w)} (A)_{(n+k+r)+(1-m)(i+j+v)+(m-1)(s+l+w)}$$

$$\times \frac{H_{n-mi,m}(x,A)H_{k-mj,m}(y,A)H_{r-mv,m}(z,A)(-iI)_s(-jI)_l(-vI)_w}{(-1)^{s+l+w} i!\,j!\,v!\,s!\,l!\,w!} t^n s^k u^r$$

Then

$$P^{A}_{n,m,k,r}(a,b,c;x,y,z) = a^n b^k c^r \left(\sqrt{2A}\right)^{-(n+k+r)} \sum_{i=0}^{[n/m]} \sum_{j=0}^{[k/m]} \sum_{v=0}^{[r/m]} \frac{(-1)^{i+j+v}}{i!\,j!\,v!}$$

$$\times (A)_{(n+k+r)+(1-m)(i+j+v)} \sum_{s=0}^{i} \sum_{l=0}^{j} \sum_{w=0}^{v} \frac{(-iI)_s(-jI)_l(-vI)_w}{s!\,l!\,w!}$$

$$\times (A+(n+k+r)I+(1-m)(i+j+v)I)_{(m-1)(s+l+w)}$$

$$\times H_{n-mi,m}(x,A)H_{k-mj,m}(y,A)H_{r-mv,m}(z,A)$$

$$P^{A}_{n,m,k,r}(a,b,c;x,y,z) = a^n b^k c^r \left(\sqrt{2A}\right)^{-(n+k+r)} \sum_{i=0}^{[n/m]} \sum_{j=0}^{[k/m]} \sum_{v=0}^{[r/m]} \frac{(-1)^{i+j+v}}{i!\,j!\,v!}$$

$$\times (A)_{(n+k+r)+(1-m)(i+j+v)} \sum_{s=0}^{i} \sum_{l=0}^{j} \sum_{w=0}^{v} \frac{(-iI)_s(-jI)_l(-vI)_w}{s!\,l!\,w!} (m-1)^{(m-1)(s+l+w)}$$

$$\times \prod_{p=1}^{m-1} \left(\frac{A+(n+k+r)+(1-m)(i+j+v)+p-1}{m-1} \right)_{(s+l+w)}$$

$$\times H_{n-mi,m}(x,A)H_{k-mj,m}(y,A)H_{r-mv,m}(z,A)$$

$$P^{A}_{n,m,k,r}(a,b,c;x,y,z)$$

$$= a^n b^k c^r \left(\sqrt{2A}\right)^{-(n+k+r)} \sum_{i=0}^{[n/m]} \sum_{j=0}^{[k/m]} \sum_{v=0}^{[r/m]} \frac{(-1)^{i+j+v}}{i!\,j!\,v!} (A)_{(n+k+r)+(1-m)(i+j+v)}$$

$$\times {}_mF_0^{(3)}\Big[-iI, -jI, -vI, \frac{A+(n+k+r)I+(1-m)(i+j+v)I}{m-1}, \ldots,$$

$$\frac{A+(n+k+r)I+(1-m)(i+j+v)I+(m-2)I}{m-1}; \ldots; (m-1)^{m-1} \Big]$$

$$\times H_{n-mi,m}(x,A)H_{k-mj,m}(y,A)H_{r-mv,m}(z,A)$$

Conclusion

In this work, we have mentioned the Humbert matrix polynomials of three variables. Hypergeometric representations, some basic relations involving the Humbert matrix polynomials, such as generating function and expansions in series of matrix polynomials.

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